

# On generalized universal irrational rotation algebras and the operator $u + v$

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## Abstract

We introduce a class of generalized universal irrational rotation  $C^*$ -algebras  $A_{\theta,\gamma} = C^*(x, w)$  which is characterized by the relations  $w^*w = ww^* = 1$ ,  $x^*x = \gamma(w)$ ,  $xx^* = \gamma(e^{-2\pi i\theta}w)$ , and  $xw = e^{-2\pi i\theta}wx$ , where  $\theta$  is an irrational number and  $\gamma(z) \in C(\mathbb{T})$  is a positive function. We characterize tracial linear functionals, simplicity, and  $K$ -groups of  $A_{\theta,\gamma}$  in terms of zero points of  $\gamma(z)$ . We show that if  $A_{\theta,\gamma}$  is simple then  $A_{\theta,\gamma}$  is an  $AT$ -algebra of real rank zero. We classify  $A_{\theta,\gamma}$  in terms of  $\theta$  and zero points of  $\gamma(z)$ . Let  $A_\theta = C^*(u, v)$  be the universal irrational rotation  $C^*$ -algebra with  $vu = e^{2\pi i\theta}uv$ . Then  $C^*(u + v) \cong A_{\theta, |1+z|^2}$ . As an application, we show that  $C^*(u + v)$  is a proper simple  $C^*$ -subalgebra of  $A_\theta$  which has a unique trace,  $K_1(C^*(u + v)) \cong \mathbb{Z}$ , and there is an order isomorphism of  $K_0(C^*(u + v))$  onto  $\mathbb{Z} + \mathbb{Z}\theta$ . Moreover,  $C^*(u + v)$  is a unital simple  $AT$ -algebra of real rank zero. We also calculate the spectrum and the Brown measure of  $u + v$ .

## 1 Introduction

The irrational rotation  $C^*$ -algebra  $A_\theta$  has been one of most studied  $C^*$ -algebras. It is known now that  $A_\theta$  is a unital simple  $C^*$ -algebra with a unique tracial state. There is an order isomorphism of  $K_0(A_\theta)$  onto  $\mathbb{Z} + \mathbb{Z}\theta$  and  $K_1(A_\theta) \cong \mathbb{Z}^2$  ([32, 34]). Moreover,  $A_\theta$  is a unital simple  $AT$ -algebra of real rank zero [11].

Let  $u$  and  $v$  be two unitary generators of the universal irrational rotation  $C^*$ -algebra  $A_\theta$  such that  $vu = e^{2\pi i\theta}uv$ . Then  $u + v$  is an abnormal operator of  $A_\theta$  and  $C^*(u + v)$  is a proper  $C^*$ -subalgebra of  $A_\theta$ . In this paper, we study the algebraic structure of  $C^*(u + v)$  and the spectral theory of  $u + v$ . Our motivation comes from our attempt to relate the theory of strongly irreducible operators relative to  $\text{II}_1$  factors with irreducible subfactors (cf. Prop. 10.7 and the question that follows).

In fact, we study a class of generalized universal irrational rotation  $C^*$ -algebras  $A_{\theta,\gamma} = C^*(x, w)$ ,

which is the universal  $C^*$ -algebra satisfying the following properties:

$$w^*w = ww^* = 1, \quad (1.1)$$

$$x^*x = \gamma(w), \quad (1.2)$$

$$xx^* = \gamma(e^{-2\pi i\theta}w), \quad (1.3)$$

$$xw = e^{-2\pi i\theta}wx, \quad (1.4)$$

where  $\theta \in (0, 1)$  and  $\gamma(z) \in C(\mathbb{T})$  is a positive continuous function of the unit circle  $\mathbb{T}$ . As we will see that  $C^*(u + v) \cong A_{\theta, |1+z|^2}$ . If  $\theta$  is an irrational number and  $\gamma(z) \equiv 1$ , then  $A_{\theta, \gamma}$  is the irrational rotation  $C^*$ -algebra  $A_\theta$ . In fact, if  $\gamma$  is invertible, then  $A_{\theta, \gamma} = A_\theta$ . However, the main interest of this paper is to study  $A_{\theta, \gamma}$  when the set of the zero points of  $\gamma(z)$  is nonempty.

It turns out that, when  $\theta$  is fixed, the  $C^*$ -algebra  $A_{\theta, \gamma}$  only depends on the set of zero points and therefore the algebraic property of  $A_{\theta, \gamma}$  is completely determined by the zero points of  $\gamma(z)$ . For example, we characterize simplicity and uniqueness of trace of  $A_{\theta, \gamma}$  as follows. Let  $Y$  be the set of zero points of  $\gamma(z)$  and let  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  be the rotation of the unit circle determined by  $\theta$ , i.e.,  $\phi(z) = e^{2\pi i\theta}z$ . Denote by  $Orb(\xi) = \{\phi^n(\xi) : n \in \mathbb{Z}\}$  for  $\xi \in \mathbb{T}$ . Then the following properties are equivalent:

1.  $A_{\theta, \gamma}$  is simple;
2.  $A_{\theta, \gamma}$  has a unique tracial state;
3.  $\phi^n(Y) \cap Y = \emptyset$  for all integer  $n \neq 0$ ;
4. For each  $\xi \in \mathbb{T}$ ,  $Orb(\xi) \cap Y$  contains at most one point.

If  $Y$  is not empty, then  $K_1(A_{\theta, \gamma}) \cong \mathbb{Z}$  and  $K_0(A_{\theta, \gamma})$  is determined by the following splitting exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow K_0(A_{\theta, \gamma}) \rightarrow C(Y, \mathbb{Z}) \rightarrow 0.$$

We also show that if  $A_{\theta, \gamma}$  is simple, then  $A_{\theta, \gamma}$  has tracial rank zero and is an inductive limit of recursive subhomogeneous  $C^*$ -algebras. As a result, the classification of  $A_{\theta, \gamma}$  falls into Elliott's classification program. Indeed, we obtain the following result. Let  $\theta_1$  and  $\theta_2$  be two irrational numbers,  $\gamma_1$  and  $\gamma_2 \in C(\mathbb{T})$  be non-negative functions and let  $Y_i$  be the set of zeros of  $\gamma_i$ ,  $i = 1, 2$ . Let  $\phi_1, \phi_2 : \mathbb{T} \rightarrow \mathbb{T}$  be rotations of the unit circle determined by  $\theta_1$  and  $\theta_2$  respectively. Suppose that  $\phi^n(Y_i) \cap Y_i = \emptyset$  for all integers  $n \neq 0$ ,  $i = 1, 2$ . Then  $A_{\theta_1, \gamma_1} \cong A_{\theta_2, \gamma_2}$  if and only if the following hold:

$$\theta_1 = \pm\theta_2 \text{ mod } (\mathbb{Z}) \text{ and } C(Y_1, \mathbb{Z})/\mathbb{Z} \cong C(Y_2, \mathbb{Z})/\mathbb{Z}.$$

In particular, when  $\gamma_1$  has only finitely many zeros, then  $A_{\theta_1, \gamma_1} \cong A_{\theta_2, \gamma_2}$  if and only if  $\theta_1 = \pm\theta_2 \text{ mod } (\mathbb{Z})$  and  $\gamma_2$  has the same number of zeros as those of  $\gamma_1$ .

A special case of interest is

$$C^*(u + v) = C^*(u + v, u^*v) = C^*(u(1 + u^*v), u^*v) \cong A_{\theta, \gamma},$$

where  $\gamma(z) = |1 + z|^2$ . As an application of the above results of generalized universal irrational rotation  $C^*$ -algebras, we show that  $C^*(u + v)$  is a proper simple  $C^*$ -subalgebra of  $A_\theta$  which has a unique trace,  $K_1(C^*(u + v)) \cong \mathbb{Z}$ , and there is an order isomorphism of  $K_0(C^*(u + v))$  onto  $\mathbb{Z} + \mathbb{Z}\theta$ . Moreover,  $C^*(u + v)$  is a unital simple  $AT$ -algebra with real rank zero. Therefore,  $C^*(u + v)$  has tracial rank zero.

The second part of the paper is to study the spectrum of  $u + v$ , which is motivated by the “the Ten Martini Problem” on the almost Mathieu operator. In mathematical physics, the almost Mathieu operator is given by

$$(H_{\lambda, \theta, \beta} u)(n) = u(n + 1) + u(n - 1) + 2\lambda \cos(2\pi(n\theta + \beta))u(n),$$

acting as a self-adjoint operator on the Hilbert space  $\ell^2(\mathbb{Z})$ . Here  $\theta, \beta, \lambda \in \mathbb{R}$  are parameters. Almost Mathieu operator was firstly introduced by R. Peierls [27] and has been extensively studied (see [22] for a recent historical account and for the physics background). In pure mathematics, its importance comes from the fact of being one of the best-understood examples of an ergodic Schrödinger operator. For example, three problems (now all solved) of Barry Simon’s fifteen problems [36] about Schrödinger operators “for the twenty-first century” featured the almost Mathieu operator. The fourth problem in [36] (known as the “the Ten Martini Problem” after Kac and Simon) conjectures that the spectrum of the almost Mathieu operator is a Cantor set for all  $\lambda \neq 0$  and irrational numbers  $\theta$ . Recently, Avila and Jitomirskaya confirmed this conjecture in [1]. For a history of this problem and earlier partial results see [22, 7, 36, 16, 8, 2, 31].

Recall that the irrational rotation  $C^*$ -algebra  $A_\theta$  can be represented on  $\ell^2(\mathbb{Z})$ , by mapping  $u$  into the bilateral shift (taking  $\phi$  into  $(\phi(n - 1))_{n \in \mathbb{Z}}$ ), and  $v$  into the operation of multiplication by  $e^{2\pi i n \theta}$  (taking  $\phi$  into  $e^{2\pi i n \theta}(\phi(n))_{n \in \mathbb{Z}}$ ), and then the polynomial  $(u + \lambda e^{2\pi i \beta} v) + (u + \lambda e^{2\pi i \beta} v)^*$  is mapped into the bounded self-adjoint operator  $H_{\lambda, \theta, \beta}$ . Since  $A_\theta$  is simple (when  $\theta$  is irrational), the spectrum of  $H_{\lambda, \theta, \beta}$  is the same as the spectrum of the element  $(u + \lambda e^{2\pi i \beta} v) + (u + \lambda e^{2\pi i \beta} v)^*$ . A natural question is that what is the spectrum of  $u + \lambda e^{2\pi i \beta} v$ ? If  $\theta$  is an irrational number, then by the uniqueness of  $A_\theta$  the spectrum of  $u + \lambda e^{2\pi i \beta} v$  is the same as  $u + |\lambda|v$ . So from now on, we always assume that  $\lambda > 0$  and  $\beta = 0$ .

Let  $\tau$  be the unique tracial state on  $A_\theta$ . By the GNS-construction, we obtain a faithful representation  $\pi$  of  $A_\theta$  on  $L^2(A_\theta, \tau)$ . The weak operator closure of  $\pi(A_\theta)$  is the hyperfinite  $\text{II}_1$  factor  $R$ . Since the spectrum of  $u + \lambda v$  is same as the spectrum of  $\pi(u + \lambda v)$  in  $R$ , we need only to calculate the

spectrum of  $\pi(u + \lambda v)$  in  $R$ . In the following we identify  $A_\theta$  with  $\pi(A_\theta)$  and thus identify  $u + \lambda v$  with  $\pi(u + \lambda v)$ .

One of the main results of the present paper is that the spectrum of  $u + \lambda v$  is given by

$$\sigma(u + \lambda v) = \begin{cases} \mathbb{T} & 0 < \lambda < 1, \\ \overline{B(0, 1)} & \lambda = 1, \\ \lambda \mathbb{T} & \lambda > 1. \end{cases}$$

Another result of spectral theory is related to the Brown measure. L. G. Brown introduced in the paper [5] a spectral distribution measure  $\mu_T$  for not necessarily normal operators  $T$  in a von Neumann algebra  $M$  with a faithful normal tracial state  $\tau$ , which is called the Brown measure of  $T$ . Recently, U. Haagerup and H. Schultz [17] proved a remarkable result about the existence of nontrivial hyperinvariant subspaces of operators in type  $\text{II}_1$  factors. They proved that if the support of  $\mu_T$  contains more than two points, then  $T$  has a nontrivial hyperinvariant space. However, the calculation of Brown measures of nonnormal operators is difficult in general (see [15, 6, 13]). In particular, Haagerup and Larsen in [15] showed that the Brown measure of the sum of two free Haar unitary operator  $T = u_1 + u_2$  is rotation invariant, has support equal to  $\overline{B(0, \sqrt{2})}$  ( $= \sigma(T)$ ), and has radial density

$$f_T(r) = \begin{cases} \frac{4}{4\pi(4-r^2)^2}, & 0 < r < \sqrt{2} \\ 0, & \text{otherwise.} \end{cases}$$

In section 12, we will show that the Brown measure of  $u + v$  (in  $R$ ) is the Haar measure on the unit circle.

This paper is organized as follows. In section 2 we introduce the class of generalized universal irrational rotation  $C^*$ -algebras  $A_{\theta, \gamma} = C^*(x, w)$ . We prove that, in fact,  $A_{\theta, \gamma}$  can be viewed as a  $C^*$ -subalgebra of  $A_\theta$ . We also fix some notation that will be used in the later sections. In section 3, we give some descriptions of the tracial state space of  $A_{\theta, \gamma}$  in terms of zero points of  $\gamma(z)$ . In particular, we show that  $A$  has a unique tracial state if and only if each rotation orbit contains at most one zero point of  $\gamma$ . In section 4, we characterize simplicity of  $A_{\theta, \gamma}$  in terms of zero points of  $\gamma(z)$ . We show that  $A_{\theta, \gamma}$  is simple if and only if it has a unique tracial state which is also equivalent to the condition that each rotation orbit contains at most one zero point of  $\gamma$ . In section 5, we construct Rieffel's projections in every simple generalized universal irrational rotation algebra  $A_{\theta, \gamma}$ . In section 6, we calculate  $K$ -groups of  $A_{\theta, \gamma}$ . In section 7, using results of section 3-6 and recent development in the Elliott's classification program, we show that when  $A_{\theta, \gamma}$  is simple, then  $A_{\theta, \gamma}$  is an  $AT$ -algebra of real rank zero. We obtain a classification result of simple  $C^*$ -algebras of  $A_{\theta, \gamma}$  in terms  $\theta$  and zero points of  $\gamma(z)$ . In section 8 we prove that the von Neumann subalgebra generated by  $u + \lambda v$  is  $R$  for

all  $0 < \lambda < \infty$ , and the  $C^*$ -algebra generated by  $u + \lambda v$  is  $C^*(u, v)$  if  $\lambda \neq 1$ . However, for  $\lambda = 1$ ,  $C^*(u + v)$  is isomorphic to  $A_{\theta, |1+z|^2}$ . Therefore  $C^*(u + v)$  is a unital simple AT-algebra of real rank zero which has  $K_1(C^*(u + v)) \cong \mathbb{Z}$  and  $K_0(u + v)$  is order isomorphic to  $\mathbb{Z} + \mathbb{Z}\theta$ . In particular,  $C^*(u + v)$  is not  $*$ -isomorphic to  $C^*(u, v)$ .

In section 9 we show that the spectral radius of  $u + \lambda v$  is 1 if  $0 < \lambda \leq 1$ . A key idea in the calculation is using Birkhoff's Ergodic theorem. Then in section 10 we show that the relative commutant of  $u + v$  in  $R$  does not contain any nontrivial idempotent. By the Riesz spectral decomposition theorem, the spectrum of  $u + v$  is connected. Combining the fact that the spectrum of  $u + v$  is rotation symmetric, in section 11 we obtain that  $\sigma(u + v) = \overline{B(0, 1)}$ . We show that the spectral radius of  $(u + \lambda v)^{-1}$  is less or equal than 1 for  $0 < \lambda < 1$ , which implies that  $\sigma(u + \lambda v)$  is contained in the unit circle  $\mathbb{T}$ . Since the spectrum of  $u + \lambda v$  is rotation symmetric,  $\sigma(u + \lambda v) = \mathbb{T}$ . By the symmetry of  $u$  and  $v$ ,  $\sigma(u + \lambda v) = \lambda \sigma(\lambda^{-1}u + v) = \lambda \mathbb{T}$  for  $\lambda > 1$ . In section 12, we calculate Brown measure of  $u + \lambda v$ .

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## 2 Generalized universal irrational rotation $C^*$ -algebras

Let  $u$  and  $v$  be two unitary generators of the universal irrational rotation  $C^*$ -algebra  $A_\theta$  such that  $vu = e^{2\pi i\theta}uv$ . To study the properties of  $C^*$ -algebras generated by  $u + v$ , we will consider the universal  $C^*$ -algebra satisfying the following properties:

$$w^*w = ww^* = 1, \quad (2.1)$$

$$x^*x = \gamma(w), \quad (2.2)$$

$$xx^* = \gamma(e^{-2\pi i\theta}w), \quad (2.3)$$

$$xw = e^{-2\pi i\theta}wx, \quad (2.4)$$

where  $\gamma(z) \in C(\mathbb{T})$  is a positive function.

A  $C^*$ -algebra  $A_{\theta, \gamma}$  is universal for the above relations provided that it is generated by operators  $x, w$  satisfying (2.1)-(2.4) and whenever  $\mathfrak{A} = C^*(x', w')$  is another  $C^*$ -algebra satisfying (2.1)-(2.4), there is a homomorphism of  $A_{\theta, \gamma}$  onto  $\mathfrak{A}$  which carries  $x$  to  $x'$  and  $w$  to  $w'$ . By (2.1),  $w$  is a unitary operator. So (2.2) implies that  $\|x\| \leq \|\gamma\|^{1/2}$ . We may consider the collection of all operators  $x_\alpha, w_\alpha$  in  $B(H_\alpha)$  satisfying (2.1)-(2.4). Then form the operator

$$x = \sum \oplus x_\alpha \quad \text{and} \quad w = \sum \oplus w_\alpha.$$

Let  $A_{\theta,\gamma} = C^*(x, w)$ . Then  $A_{\theta,\gamma}$  is the desired universal algebra. Note that if  $\gamma(z) \equiv 1$ , then  $A_{\theta,\gamma}$  is precisely the universal irrational rotation algebra. So we call  $A_{\theta,\gamma}$  a *generalized universal irrational rotation algebra*.

Let  $A_\theta$  be the universal irrational rotation  $C^*$ -algebra with two unitary generators  $u, v$  with  $vu = e^{2\pi i\theta}uv$ . Then  $u\gamma(v)^{1/2}$  and  $v$  satisfy (2.1)-(2.4). So there is a  $*$ -homomorphism from  $A_{\theta,\gamma}$  onto the  $C^*$ -subalgebra of  $A_\theta$  generated by  $u\gamma(v)^{1/2}$  and  $v$ . We will show that we may view  $A_{\theta,\gamma}$  as the  $C^*$ -subalgebra of  $A_\theta$  generated by  $u\gamma(v)^{1/2}$  and  $v$  and  $C^*(u + v) \cong A_{\theta,|1+z|^2}$ .

By (2.1)-(2.4) and simple calculations, we have the following equations.

$$x^*w = e^{2\pi i\theta}wx^*, \quad (2.5)$$

$$xf(w) = f(e^{-2\pi i\theta}w)x, \quad \forall f(z) \in C(\mathbb{T}), \quad (2.6)$$

$$x^*f(w) = f(e^{2\pi i\theta}w)x^*, \quad \forall f(z) \in C(\mathbb{T}), \quad (2.7)$$

$$(x^*)^r x^r = \gamma(e^{2\pi i(r-1)\theta}w) \gamma(e^{2\pi i(r-2)\theta}w) \cdots \gamma(w), \quad (2.8)$$

$$x^r (x^*)^r = \gamma(e^{-2\pi ir\theta}w) \gamma(e^{-2\pi i(r-1)\theta}w) \cdots \gamma(e^{-2\pi i\theta}w). \quad (2.9)$$

We apply the universal property to obtain certain special automorphisms of  $A_{\theta,\gamma}$ . For any constant  $\lambda = e^{2\pi it}$  on the unit circle, the pair  $(\lambda x, w)$  also satisfy (2.1)-(2.4). Thus there is an endomorphism of  $A_{\theta,\gamma}$  such that  $\rho_t(x) = \lambda x$  and  $\rho_t(w) = w$ . By symmetry,  $\rho_{-t}(x) = \bar{\lambda}x$  and  $\rho_{-t}(w) = w$ . Hence,  $\rho_{-t}(\rho_t(x)) = \rho_t(\rho_{-t}(x)) = x$  and  $\rho_{-t}(\rho_t(w)) = \rho_t(\rho_{-t}(w)) = w$ . This implies that  $\rho_t$  is an automorphism of  $A_{\theta,\gamma}$ .

For each fixed  $a$  in  $A_{\theta,\gamma}$ , the map from  $[0, 1]$  to  $A_{\theta,\gamma}$  given by  $f(t) = \rho_t(a)$  is norm continuous. To verify this, notice that it is true for all noncommutative polynomials in  $x, x^*, w, w^*$ . These are dense and automorphisms are contractive; so the rest follows from a simple approximation argument.

Define a map of  $A_{\theta,\gamma}$  into itself by

$$\Phi(a) = \int_0^1 \rho_t(a) dt.$$

Then the integral makes sense as Riemann sum because the integrand is a norm continuous function. By (2.1)-(2.9) and simple calculations, we can see that the following set

$$\left\{ \sum_{n=1}^N x^n f_n(w) + f_0(w) + \sum_{n=1}^N f_{-n}(w)(x^*)^n \mid N \in \mathbb{N}, f_n(z), f_{-n}(z) \in C(\mathbb{T}) \right\}$$

is dense in  $A_{\theta,\gamma}$ .

The proof of the following proposition is similar to the proof of Theorem VI.1.1 of [9]. For the sake of completeness, we include a detailed proof.

**Proposition 2.1.** *The map  $\Phi$  is a faithful conditional expectation of  $A_{\theta,\gamma}$  onto  $C^*(w)$  such that  $\Phi(x^k f(w)) = \Phi(f(w)(x^*)^k) = 0$  for all  $f(z) \in C(\mathbb{T})$  and  $k \in \mathbb{N}$ . In addition, for every  $a \in A_{\theta,\gamma}$ ,*

$$\Phi(a) = \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{j=-n}^n w^j a (w^*)^j.$$

*Proof.* Since,  $\|\rho_t(a)\| = \|a\|$ ,

$$\left\| \sum_{j=1}^n \rho_{t_j}(a) \beta_j \right\| \leq \|a\|$$

for any scalar  $0 \leq \beta_j \leq 1$  such that  $\sum_{j=1}^n \beta_j = 1$ . It follows that

$$\|\Phi(a)\| = \left\| \int_0^1 \rho_t(a) dt \right\| \leq \|a\|.$$

We conclude that  $\|\Phi\| \leq 1$ . Since  $\Phi(1) = 1$ ,  $\|\Phi\| = 1$ . Since  $\rho_t(w) = w$  for all  $t$ ,  $\rho_t(a) = a$  for all  $a \in C^*(w)$ . Hence  $\Phi(a) = a$  for all  $a \in C^*(w)$ . By the definition of  $\Phi$ ,

$$\Phi(a_1 a a_2) = \int_0^1 \rho_t(a_1 a a_2) dt = \int_0^1 \rho_t(a_1) \rho_t(a) \rho_t(a_2) dt = \int_0^1 a_1 \rho_t(a) a_2 dt = a_1 \Phi(a) a_2$$

for all  $a_1, a_2 \in C^*(w)$  and  $a \in A_{\theta,\gamma}$ .

Suppose  $a = x^k f(w)$  for  $f(z) \in C(\mathbb{T})$  and  $k \in \mathbb{N}$ . Then

$$\Phi(a) = \int_0^1 \rho_t(x^k f(w)) dt = \int_0^1 \rho_t(x^k) \rho_t(f(w)) dt = \int_0^1 e^{2\pi i k t} x^k f(w) dt = \left( \int_0^1 e^{2\pi i k t} dt \right) a = 0.$$

Suppose  $a = f(w)(x^*)^k$  for  $f(z) \in C(\mathbb{T})$  and  $k \in \mathbb{N}$ . Then

$$\Phi(a) = \int_0^1 \rho_t(f(w)(x^*)^k) dt = \int_0^1 \rho_t(f(w)) \rho_t((x^*)^k) dt = \int_0^1 e^{-2\pi i k t} f(w)(x^*)^k dt = \left( \int_0^1 e^{-2\pi i k t} dt \right) a = 0.$$

Since  $\|\Phi\| = 1$ ,  $\Phi(A_{\theta,\gamma}) \subseteq C^*(w)$ . By Tomiyama's Theorem [37],  $\Phi$  is a conditional expectation of  $A_{\theta,\gamma}$  onto  $C^*(w)$ . If  $a$  is positive and nonzero, then  $\rho_t(a)$  is positive and nonzero for all  $t$ . Thus the integral  $\Phi(a)$  is positive and nonzero. Hence  $\Phi$  is faithful.

Suppose  $a = x^k f(w)$  for  $f(z) \in C(\mathbb{T})$  and  $k \in \mathbb{N}$ . By equations (2.4) and (2.5),

$$\lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{j=-n}^n w^j a (w^*)^j = \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{j=-n}^n e^{2\pi i j k \theta} a = \lim_{n \rightarrow \infty} \frac{1}{2n+1} \left( \frac{\sin(2n+1)\pi k \theta}{\sin \pi k \theta} \right) a = 0.$$

Hence

$$\Phi(a) = \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{j=-n}^n w^j a(w^*)^j = 0.$$

Similarly, we can show that if  $a = f(w)(x^*)^k$  for  $f(z) \in C(\mathbb{T})$  and  $k \in \mathbb{N}$  then

$$\Phi(a) = \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{j=-n}^n w^j a(w^*)^j = 0.$$

If  $a = f(w)$  for some  $f(z) \in C(\mathbb{T})$ , then

$$\Phi(a) = \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{j=-n}^n w^j a(w^*)^j = a.$$

By linearity and continuity, this formula is valid for all  $a$  in  $A_{\theta, \gamma}$ . □

**Corollary 2.2.**  $\forall a \in A_{\theta, \gamma}$ ,  $\rho_t(a) = a$  for all  $0 \leq t \leq 1$  if and only if  $a \in C^*(w)$ .

*Proof.* If  $a \in A_{\theta, \gamma}$  and  $\rho_t(a) = a$  for all  $0 \leq t \leq 1$ , then  $a = \Phi(a) \in C^*(w)$  by Proposition 2.1. Conversely, since  $\rho_t(w) = w$  for all  $t$ ,  $\rho_t(a) = a$  for all  $t$  and  $a \in C^*(w)$ . □

Let  $m = dz/2\pi$  be the unique Haar measure on  $\mathbb{T}$ .

**Remark 2.3.** If  $\gamma(z) \in C(\mathbb{T})$  is a positive function with  $m(\{z | \gamma(z) = 0\}) = 0$ , then (2.3) can be replaced by a weaker condition

$$xx^* \in C^*(w). \tag{2.10}$$

To see this, let  $xx^* = h(w)$  for some  $h(z) \in C(\mathbb{T})$ . Then by (2.5)

$$\gamma(w)^2 = x^*xx^*x = x^*h(w)x = h(e^{2\pi i\theta}w)x^*x = h(e^{2\pi i\theta}w)\gamma(w).$$

Hence  $\gamma(z)^2 = h(e^{2\pi i\theta}z)\gamma(z)$ . Let  $E = \{z | \gamma(z) = 0\}$ . Then for  $z \in \mathbb{T} \setminus E$ ,  $\gamma(z) = h(e^{2\pi i\theta}z)$ . Since  $m(\mathbb{T} \setminus E) = 1$ ,  $\gamma(z) = h(e^{2\pi i\theta}z)$  for all  $z \in \mathbb{T}$ . Thus  $h(z) = \gamma(e^{-2\pi i\theta}z)$ , which is (2.3).

Note that in the irrational rotation  $C^*$ -algebra  $C^*(u, v)$  with  $vu = e^{2\pi i\theta}uv$ ,  $u\gamma(v)^{1/2}$  and  $v$  satisfy (2.1)-(2.4). So there exists a homomorphism  $\varphi$  from  $A_{\theta, \gamma}$  onto  $C^*(u\gamma(v)^{1/2}, v)$  such that  $\varphi(x) = u\gamma(v)^{1/2}$  and  $\varphi(w) = v$ . Since the spectrum  $\sigma(v)$  is  $\mathbb{T}$ ,  $\sigma(w) = \mathbb{T}$ . Hence  $C^*(w) \cong C(\mathbb{T})$ . In the following, we identify  $C^*(w)$  with  $C(\mathbb{T})$ . Let  $\rho$  be the state on  $C(\mathbb{T})$  induced by the Haar measure  $m$  on  $\mathbb{T}$ . Then  $\rho$  is faithful on  $C^*(w)$ .

**Lemma 2.4.** For  $a \in A_{\theta, \gamma}$ , let  $\tau(a) = \rho \cdot \Phi(a)$ . Then  $\tau$  is a faithful trace on  $A_{\theta, \gamma}$ .



*Proof.* Since  $\rho$  is a faithful state on  $C^*(w)$  and  $\Phi$  is a faithful conditional expectation of  $A_{\theta,\gamma}$  onto  $C^*(w)$ ,  $\tau$  is a faithful state on  $A_{\theta,\gamma}$ . We only need to verify  $\tau$  is a trace. Note that the following set

$$\left\{ \sum_{n=1}^N x^n f_n(w) + f_0(w) + \sum_{n=1}^N f_{-n}(w)(x^*)^n \mid N \in \mathbb{N}, f_n(z), f_{-n}(z) \in C(\mathbb{T}) \right\}$$

is dense in  $A_{\theta,\gamma}$ . By boundedness, linearity and positivity of  $\tau$ , we need only to verify  $\tau(ab) = \tau(ba)$  for the following two cases.

Case 1.  $a = x^r f(w)$ ,  $b = x^s g(w)$ ,  $r, s \geq 0$ . If  $r + s = 0$ , i.e.,  $r = s = 0$ , then  $\tau(ab) = \tau(ba)$  is trivial. Suppose  $r + s > 0$ . Then

$$\tau(ab) = \tau(x^r f(w) x^s g(w)) = \tau(x^{r+s} f(e^{2\pi i s \theta} w) g(w)) = \rho(\Phi(x^{r+s}) f(e^{2\pi i s \theta} w) g(w)) = 0,$$

and

$$\tau(ba) = \tau(x^s g(w) x^r f(w)) = \tau(x^{r+s} g(e^{2\pi i r \theta} w) f(w)) = \rho(\Phi(x^{r+s}) g(e^{2\pi i r \theta} w) f(w)) = 0.$$

So  $\tau(ab) = \tau(ba)$ .

Case 2.  $a = x^r f(w)$ ,  $b = g(w)(x^*)^s$ ,  $r, s \geq 0$ . If  $r > s$ , then

$$\begin{aligned} \tau(ab) &= \tau(x^r f(w) g(w)(x^*)^s) = \tau(x^{r-s} f(e^{-2\pi i s \theta} w) g(e^{-2\pi i s \theta} w) x^s (x^*)^s) \\ &= \rho(\Phi(x^{r-s}) f(e^{-2\pi i s \theta} w) g(e^{-2\pi i s \theta} w) x^s (x^*)^s) = 0, \end{aligned}$$

and

$$\begin{aligned} \tau(ba) &= \tau(g(w)(x^*)^s x^r f(w)) = \tau(g(w)(x^*)^s x^s f(e^{-2\pi i (r-s) \theta} w) x^{r-s}) \\ &= \rho(g(w)(x^*)^s x^s f(e^{-2\pi i (r-s) \theta} w) \Phi(x^{r-s})) = 0. \end{aligned}$$

So  $\tau(ab) = \tau(ba)$ . Similarly, we can show that if  $r < s$  then  $\tau(ab) = \tau(ba)$ . If  $r = s$ , then we have

$$\begin{aligned} \tau(ba) &= \rho(ba) = \rho(g(w)(x^*)^r x^r f(w)) = \rho(g(w) f(w) \gamma(e^{2\pi i (r-1) \theta} w) \gamma(e^{2\pi i (r-2) \theta} w) \cdots \gamma(w)), \\ &= \int_{\mathbb{T}} f(z) g(z) \gamma(e^{2\pi i (r-1) \theta} z) \gamma(e^{2\pi i (r-2) \theta} z) \cdots \gamma(z) dm(z), \\ \tau(ab) &= \rho(ab) = \rho(x^r f(w) g(w)(x^*)^r) = \rho(f(e^{-2\pi i r \theta} w) g(e^{-2\pi i r \theta} w) x^r (x^*)^r) \\ &= \rho(f(e^{-2\pi i r \theta} w) g(e^{-2\pi i r \theta} w) \gamma(e^{-2\pi i \theta} w) \gamma(e^{-2\pi i 2 \theta} w) \cdots \gamma(e^{-2\pi i r \theta} w)) \\ &= \int_{\mathbb{T}} f(e^{-2\pi i r \theta} z) g(e^{-2\pi i r \theta} z) \gamma(e^{-2\pi i r \theta} \cdot e^{2\pi i (r-1) \theta} z) \gamma(e^{-2\pi i r \theta} \cdot e^{2\pi i (r-2) \theta} z) \cdots \gamma(e^{-2\pi i r \theta} z) dm(z). \end{aligned}$$

Since  $m$  is the Haar measure on  $\mathbb{T}$ ,  $\tau(ab) = \tau(ba)$ .

□

**Theorem 2.5.** *The homomorphism  $\varphi$  from  $A_{\theta,\gamma}$  onto  $C^*(u\gamma(v)^{1/2}, v)$  such that  $\varphi(x) = u\gamma(v)^{1/2}$  and  $\varphi(w) = v$  is an isomorphism.*

*Proof.* Consider the GNS-construction of  $A_{\theta,\gamma}$  with respect to the faithful trace  $\tau$ . Then we may assume that  $A_{\theta,\gamma}$  faithfully acts on the Hilbert space  $L^2(A_{\theta,\gamma}, \tau)$ . Let  $\tau'$  be the unique trace on  $C^*(u, v)$ , and let  $x' = u\gamma(v)$ ,  $w' = v$ . For a noncommutative polynomial  $p$  in four variables, we have  $\tau(p(x, x^*, w, w^*)) = \tau'(p(x', (x')^*, w', (w')^*))$ . Hence the operator  $U : p(x, x^*, w, w^*) \rightarrow p(x', (x')^*, w', (w')^*)$  extends to a unitary operator from  $L^2(A_{\theta,\gamma}, \tau)$  onto  $L^2(C^*(u\gamma(v)^{1/2}, v), \tau')$ . So  $\varphi(x) = U^*xU$  is an isomorphism.  $\square$

In what follows, we will identify  $A_{\theta,\gamma}$  with the  $C^*$ -subalgebra of  $A_\theta$  generated by  $u\gamma^{1/2}(v)$  and  $v$ . We will take advantage of the knowledge of  $A_\theta$  to study  $A_{\theta,\gamma}$ . We will use the following conventions:

**Definition 2.6.** We may view  $A_\theta = C(\mathbb{T}) \rtimes_\phi \mathbb{Z}$ , where  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  is defined by  $\phi(z) = e^{2\pi i\theta}z$  for all  $z \in \mathbb{T}$ . Define  $\alpha_\theta : C(\mathbb{T}) \rightarrow C(\mathbb{T})$  by  $\alpha_\theta(f) = f \circ \phi$  for all  $f \in C(\mathbb{T})$ . Denote by  $u$  the unitary in  $A_\theta$  implementing  $\alpha_\theta$ , i.e.,  $u^*fu = \alpha_\theta(f) = f \circ \phi$  for all  $f \in C(\mathbb{T})$ .

Let  $\gamma : \mathbb{T} \rightarrow R_+$  be a nonnegative continuous function and let

$$Y = \{z \in \mathbb{T} : \gamma(z) = 0\}.$$

Viewing  $A_{\theta,\gamma}$  as a  $C^*$ -subalgebra of  $A_\theta$ , it is easy to check that

$$A_{\theta,\gamma} = C^*(C(\mathbb{T}), uC_0(\mathbb{T} \setminus Y)),$$

the  $C^*$ -subalgebra of  $A_\theta$  generated by  $C(\mathbb{T})$  and  $\{uf : f \in C_0(\mathbb{T} \setminus Y)\}$ .

Let  $\xi \in \mathbb{T}$  denote by

$$\text{Orb}(\xi) = \{\phi^n(\xi) : n \in \mathbb{Z}\}$$

the orbit of  $\xi$  under the rotation  $\phi$ .

The following is an easy fact:

**Proposition 2.7.** *Let  $\theta \in (0, 1)$  be an irrational number and let  $Y \subset \mathbb{T}$  be a subset. Then the following are equivalent:*

- (1).  $\phi^n(Y) \cap Y = \emptyset$  for any integer  $n \neq 0$ ;
- (2). For each  $\xi \in \mathbb{T}$ ,  $\text{Orb}(\xi) \cap Y$  contains at most one point;
- (3).  $Y_1 \cap Y_2 = \emptyset$ , where  $Y_1 = \bigcup_{n \geq 0} \phi^n(Y)$  and  $Y_2 = \bigcup_{k \geq 1} \phi^{-k}(Y)$ .

*Proof.* (1)  $\Rightarrow$  (2): Suppose that  $\phi^{k_1}(\xi), \phi^{k_2}(\xi) \in Y$  for integers  $k_1 \neq k_2$ . Then  $\phi^{k_1}\xi \in \phi^{k_1-k_2}(Y) \cap Y$ . This is contradiction. So (2) holds.

(2)  $\Rightarrow$  (3): If  $\xi \in Y_1 \cap Y_2$ , then there are  $\xi_1, \xi_2 \in Y$  such that  $\xi = e^{2\pi i n \theta} \xi_1 = e^{-2\pi i k \theta} \xi_2$  for some  $n \geq 0$  and  $k \geq 1$ . It follows that  $\xi_1 \in Y$  and  $e^{2\pi i(n+k)\theta} \xi_1 \in Y$ . By (2), this is impossible. So (3) holds.

(3)  $\Rightarrow$  (1): Suppose that  $\xi \in \phi^n(Y) \cap Y$  for some integer  $n \neq 0$ . If  $n \leq -1$ , then  $\xi \in Y_1 \cap Y_2$ . If  $n \geq 1$ , then  $\phi^{-n}(\xi) \in Y \cap \phi^{-n}(Y) \subset Y_1 \cap Y_2$ .

□

### 3 Traces on generalized universal irrational rotation $C^*$ -algebras

We will continue to study the traces on  $A_{\theta, \gamma}$ . Here, again,  $\gamma \in C(\mathbb{T})$  is a positive function and  $Y$  is the set of zeros of  $\gamma$ . The proof of Lemma 2.4 indeed implies the following result.

**Proposition 3.1.** *If  $\mu$  is a complex regular Borel measure on  $\mathbb{T}$  which satisfies that*

$$\int_{\mathbb{T}} f(e^{-2\pi i \theta} z) d\mu(z) = \int_{\mathbb{T}} f(z) d\mu(z) \quad (3.1)$$

*for all  $f(z)$  in  $C_0(\widetilde{\mathbb{T} \setminus Y})$ , the unitization of  $C_0(\mathbb{T} \setminus Y)$ , and let  $\sigma(f) = \int_{\mathbb{T}} f(z) d\mu(z)$  for  $f(z) \in C(\mathbb{T})$ , then  $\sigma \cdot \Phi$  is a bounded tracial linear functional on  $A_{\theta, \gamma}$ . Conversely, every bounded tracial linear functional on  $A_{\theta, \gamma}$  is given in this way.*

*Proof.* If  $\mu$  satisfies (3.1) for all  $f(z)$  in  $C_0(\widetilde{\mathbb{T} \setminus Y})$ , then by a similar argument of the proof of Lemma 2.4,  $\sigma \cdot \Phi$  is a bounded tracial linear functional on  $A_{\theta, \gamma}$ . Conversely, suppose  $\sigma$  is a bounded tracial linear functional on  $A_{\theta, \gamma}$ . By Proposition 2.1,  $\sigma(a) = \sigma(\Phi(a))$ . By the Riesz representation theorem,  $\sigma$  induces a complex regular Borel measure  $\mu$  on  $\mathbb{T}$ .

Since for all  $f(z) \in C(\mathbb{T})$ ,

$$\begin{aligned} \sigma(xf(w)\overline{f(w)}x^*) &= \sigma(|f|^2(e^{-2\pi i \theta} w)\gamma(e^{-2\pi i \theta} w)) \\ &= \int_{\mathbb{T}} |f|^2(e^{-2\pi i \theta} z)\gamma(e^{-2\pi i \theta} z) d\mu(z) = \int_{\mathbb{T}} |f|^2(z)\gamma(z) d\mu(e^{2\pi i \theta} z) \end{aligned}$$

and

$$\sigma(\overline{f(w)}x^*xf(w)) = \sigma(|f|^2(w)\gamma(w)) = \int_{\mathbb{T}} |f|^2(z)\gamma(z) d\mu(z),$$

we have

$$\int_{\mathbb{T}} |f|^2(z)\gamma(z) d\mu(e^{2\pi i \theta} z) = \int_{\mathbb{T}} |f|^2(z)\gamma(z) d\mu(z), \quad \forall f(z) \in C(\mathbb{T}).$$

Since every continuous function is a linear combinations of positive functions,

$$\int_{\mathbb{T}} f(z)\gamma(z) d\mu(e^{2\pi i \theta} z) = \int_{\mathbb{T}} f(z)\gamma(z) d\mu(z), \quad \forall f(z) \in C(\mathbb{T}).$$

This implies that (3.1) is true for all  $f(z) \in \overline{\gamma(z)C(\mathbb{T})} = C_0(\mathbb{T} \setminus Y)$ . Since (3.1) is true for  $f(z) \equiv 1$ ,  $\mu$  is a regular Borel measure on  $\mathbb{T}$  which satisfies

$$\int_{\mathbb{T}} f(e^{-2\pi i\theta} z) d\mu(z) = \int_{\mathbb{T}} f(z) d\mu(z)$$

for all  $f(z)$  in  $\widetilde{C_0(\mathbb{T} \setminus Y)}$ .

□

Recall that  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  is the rotation of circle by  $\theta$ , i.e.,  $\phi(z) = e^{2\pi i\theta} z$  for  $z \in \mathbb{T}$ .

**Theorem 3.2.** *Let  $Y$  be the set of zero points of  $\gamma(z)$ . Then the following conditions are equivalent:*

1. *There exists a unique trace on  $A_{\theta, \gamma}$ ;*
2.  *$\phi^n(Y) \cap Y = \emptyset$  for all integers  $n \neq 0$ ;*
3. *For each  $\xi \in \mathbb{T}$ ,  $\text{Orb}(\xi) \cap Y$  contains at most one point.*

*Proof.* The equivalence of 2 and 3 follows from Proposition 2.7.

“1  $\Rightarrow$  2”. Suppose that  $\phi^k(Y) \cap Y \neq \emptyset$  for some integers  $k \neq 0$ . Assume that  $z_1 \in Y$  and  $z_2 = \phi^k(z_1) = e^{2\pi i k \theta} z_1 \in Y$ . By symmetry, we may assume that  $k > 0$ . Let

$$\mu = \frac{\delta_{e^{2\pi i\theta} z_1} + \delta_{e^{2\pi i 2\theta} z_1} + \cdots + \delta_{z_2}}{k},$$

where  $\delta_t$  is the point-mass at  $t$ . Then  $\widetilde{C_0(\mathbb{T} \setminus Y)} \subseteq \{f \in C(\mathbb{T}) : f(z_1) = f(z_2)\}$ . Note that for  $f(z) \in C(\mathbb{T})$  with  $f(z_1) = f(z_2)$  we have

$$\begin{aligned} \int_{\mathbb{T}} f(e^{-2\pi i\theta} z) d\mu(z) &= \frac{f(z_1) + f(e^{2\pi i\theta} z_1) + \cdots + f(e^{2\pi i(k-1)\theta} z_1)}{k} \\ &= \frac{f(e^{2\pi i\theta} z_1) + \cdots + f(e^{2\pi i(k-1)\theta} z_1) + f(z_2)}{k} = \int_{\mathbb{T}} f(z) d\mu(z). \end{aligned}$$

By Proposition 3.1,  $\mu$  induces a trace different from the trace given in Lemma 2.4.

“2  $\Rightarrow$  1” Let  $C = \widetilde{C_0(\mathbb{T} \setminus Y)}$  be the unitization of  $C_0(\mathbb{T} \setminus Y)$ , and let  $\rho$  be a tracial state on  $A_{\theta, \gamma}$ . It follows from 3.1 that  $\rho = \mu \circ \Phi$ , where  $\mu$  is a Borel probability measure on  $\mathbb{T}$  such that

$$\int_{\mathbb{T}} f(\phi^{-1}(z)) d\mu(z) = \int_{\mathbb{T}} f(z) d\mu(z) \tag{3.2}$$

for all  $f \in C$ . Define  $X_0 = Y$  and  $X_n = \phi^n(Y)$ ,  $n = \pm 1, \pm 2, \dots$ . By the assumption,  $\{X_n : n \in \mathbb{Z}\}$  are mutually disjoint closed subsets of  $\mathbb{T}$ . We claim that

$$\mu(X_n) = 0, \quad n \in \mathbb{Z}. \tag{3.3}$$

Let  $k \geq 1$  be an integer. One can find an open subset  $G \subset \mathbb{T}$  such that

$$X_0 \subset G \text{ and } \phi^j(G) \cap \phi^i(G) = \emptyset \quad (3.4)$$

if  $i \neq j$  and  $-k \leq i, j \leq k$ . Define  $0 \leq g \leq 1$  in  $C(\mathbb{T})$  such that  $g(z) = 0$  if  $z \in X_0$  and  $g(z) = 1$  if  $z \in \mathbb{T} \setminus G$ . Then  $g \in C_0(\mathbb{T} \setminus Y)$ . Let  $h = 1 - g$ . Then  $h(z) = 1$  if  $z \in X_0$  and  $h(z) = 0$  if  $z \in \mathbb{T} \setminus G$ . Moreover,  $h \in C$ . Let  $h_j = h \circ \phi^{-j}$ ,  $-k \leq j \leq k$ . Note that  $h_j(z) = 1$  if  $z \in \phi^j(X_0)$  and  $h_j(z) = 0$  if  $z \in \mathbb{T} \setminus \phi^j(G)$  for  $-k \leq j \leq k$ . In particular, if  $-k \leq j \leq k$  and  $j \neq 0$ , then  $h_j(z) = 0$  for  $z \in X_0$ . Therefore  $h_j \in C_0(\mathbb{T} \setminus Y) \subset C$  for  $-k \leq j \leq k$  and  $j \neq 0$ . It follows from 3.2 that

$$\int_{\mathbb{T}} h_j d\mu = \int_{\mathbb{T}} h d\mu, \quad -k \leq j \leq k. \quad (3.5)$$

Since  $h_j$  has disjoint support, (3.5) implies that

$$0 \leq \int_{\mathbb{T}} h_j d\mu = \int_{\mathbb{T}} h d\mu < \frac{1}{2k+1}. \quad (3.6)$$

Therefore,

$$\mu(X_j) < \frac{1}{2k+1}, \quad -k \leq j \leq k. \quad (3.7)$$

Since (3.7) holds for any integer  $k \geq 1$ , we conclude that the claim (3.3) holds.

Let  $f \in C(\mathbb{T})$  and let  $\epsilon > 0$ . Since  $\mu(X_0) = \mu(X_{-1}) = 0$ , we can choose an open subset  $O \subset \mathbb{T}$  such that

$$Y \subset O, \mu(O) < \epsilon/(2\|f\| + 1) \text{ and } \mu(\phi^{-1}(O)) < \epsilon/(2\|f\| + 1). \quad (3.8)$$

Define a continuous function  $g_1 \in C$  such that  $0 \leq g_1 \leq 1$ ,

$$g_1(z) = 0, \text{ when } z \in Y \text{ and } g_1(z) = 1 \text{ when } z \in \mathbb{T} \setminus O. \quad (3.9)$$

Note that  $fg_1 \in C$ . In particular,

$$\int_{\mathbb{T}} fg_1 \circ \phi^{-1} d\mu = \int_{\mathbb{T}} fg_1 d\mu. \quad (3.10)$$

Then

$$\left| \int_{\mathbb{T}} f(e^{-2i\pi\theta} z) d\mu(z) - \int_{\mathbb{T}} f(z) d\mu(z) \right| \quad (3.11)$$

$$\leq \left| \int_{\mathbb{T}} (f - g_1 f) \circ \phi^{-1} d\mu \right| + \left| \int_{\mathbb{T}} (fg_1 - fg_1 \circ \phi^{-1}) d\mu \right| \quad (3.12)$$

$$+ \left| \int_{\mathbb{T}} (f - g_1 f) d\mu \right| \quad (3.13)$$

$$\leq \int_{\mathbb{T}} |(f - g_1 f) \circ \phi^{-1}| d\mu + \int_{\mathbb{T}} |f - g_1 f| d\mu \quad (3.14)$$

$$\leq \|f\| \mu(\phi^{-1}(O)) + \|f\| \mu(O) < \epsilon/2 + \epsilon/2 = \epsilon \quad (3.15)$$

It follows that

$$\int_{\mathbb{T}} f(e^{-2i\pi\theta} z) d\mu(z) = \int_{\mathbb{T}} f(z) d\mu(z) \quad (3.16)$$

for all  $f \in C(\mathbb{T})$ . Therefore,  $\mu$  is the Haar measure on  $\mathbb{T}$ . This shows that  $A_{\theta,\gamma}$  has a unique tracial state.  $\square$

**Remark 3.3.** If  $\gamma(z)$  has a single zero point, then there exists a unique tracial state on  $A_{\theta,\gamma}$

**Remark 3.4.** If  $\gamma(z)$  has two zero points  $z_1, z_2$ , then there exists a unique tracial state on  $A_{\theta,\gamma}$  if and only if there does not exist  $k \in \mathbb{N}$  such that  $z_2 = e^{2\pi i k \theta} z_1$ .

For a  $C^*$ -algebra  $\mathfrak{A}$ , we denote by  $\text{Tr}(\mathfrak{A})$  the space of bounded tracial linear functionals on  $\mathfrak{A}$ . Denote by  $T(\mathfrak{A})$  the tracial state space of  $\mathfrak{A}$ .

Let  $\Delta$  be a subset of  $\mathbb{T}$  which contains exactly one point of each orbit  $\text{Orb}(\xi)$  and let  $Y$  be the set of zeros of  $\gamma(z)$ .

**Lemma 3.5.** Let  $\xi_1, \xi_2, \dots, \xi_r \in \Delta$  and  $Y_j = Y \cap \text{Orb}(\xi_j)$ ,  $j = 1, 2, \dots, r$ . Let  $Y'_j \subset Y_j$  be a finite subset of  $Y_j$  and let  $|Y'_j|$  be the cardinality of  $Y'_j$ . Then  $\dim \text{Tr}(A_{\theta,\gamma}) \geq 1 + \sum_{j=1}^r (|Y'_j| - 1)$ .

*Proof.* Suppose that  $Y'_j = \{z_{j,1}, (e^{2\pi i m_{j,1}\theta})z_{j,1}, \dots, (e^{2\pi i m_{j,n_j}\theta})z_{j,1}\}$  with  $1 < m_{j,1} < \dots < m_{j,n_j}$ , where  $|Y'_j| = n_j + 1$ ,  $j = 1, 2, \dots, r$ . As in the proof of Proposition 3.2, the Haar measure  $m$  together with

$$\begin{aligned} \mu_{j,1} &= \frac{\delta_{(e^{2\pi i \theta})z_{j,1}} + \dots + \delta_{(e^{2\pi i m_{j,1}\theta})z_{j,1}}}{m_{j,1} - 1} \\ \mu_{j,2} &= \frac{\delta_{(e^{2\pi i (m_{j,1}+1)\theta})z_{j,1}} + \dots + \delta_{(e^{2\pi i m_{j,2}\theta})z_{j,1}}}{m_{j,2} - m_{j,1}} \\ &\vdots \\ \mu_{j,n_j} &= \frac{\delta_{(e^{2\pi i (m_{j,n_j-1}+1)\theta})z_{j,1}} + \dots + \delta_{(e^{2\pi i m_{j,n_j}\theta})z_{j,1}}}{m_{n_j} - m_{n_j-1}} \end{aligned}$$

induce  $1 + \sum_{j=1}^r (|Y'_j| - 1)$  linearly independent tracial states on  $A_{\theta,\gamma}$ . This proves that  $\dim(\text{Tr}(A_{\theta,\gamma})) \geq 1 + \sum_{j=1}^r (|Y'_j| - 1)$ .  $\square$

**Corollary 3.6.** Let  $\xi \in \mathbb{T}$  and let  $N(\xi)$  be the number of points in  $Y \cap \text{Orb}(\xi)$ . If  $\sum_{\xi \in \Delta} N(\xi) = \infty$ , then  $A_{\theta,\gamma}$  has infinitely many extreme points in its tracial state space  $T(A_{\theta,\gamma})$  and  $\dim(\text{Tr}(A_{\theta,\gamma})) = \infty$ .

*Proof.* For any integer  $N \geq 1$ , since  $\sum_{\xi \in \Delta} N(\xi) = \infty$ , one can find  $\xi_1, \xi_2, \dots, \xi_n \in \mathbb{T}$  and finite subsets  $Y_j \subset Y \cap \text{Orb}(\xi_j)$ ,  $j = 1, 2, \dots, n$  such that

$$\sum_{j=1}^n (|Y_j| - 1) > N.$$

It follows from 3.5 that  $\dim(\text{Tr}(A_{\theta, \gamma})) \geq N$ . It follows that  $\dim(\text{Tr}(A_{\theta, \gamma})) = \infty$ . The corollary follows.  $\square$

**Proposition 3.7.** *Let  $\xi_1, \xi_2, \dots, \xi_r \in \Delta$  and  $Y_j = Y \cap \text{Orb}(\xi_j)$ ,  $j = 1, 2, \dots, r$ , such that  $Y = \cup_{j=1}^r Y_j$ . Suppose that  $Y$  is a finite set. Then  $\dim(\text{Tr}(A_{\theta, \gamma})) = 1 + \sum_{j=1}^r (|Y_j| - 1)$ , where  $|Y_j|$  is the number of elements in  $Y_j$ .*

*Proof.* By Lemma 3.5,  $\dim(\text{Tr}(A_{\theta, \gamma})) \geq 1 + \sum_{j=1}^r (|Y_j| - 1)$ . We need to show  $\dim(\text{Tr}(A_{\theta, \gamma})) \leq 1 + \sum_{j=1}^r (|Y_j| - 1)$ . By Proposition 3.1, a regular Borel probability measure  $\mu$  on  $\mathbb{T}$  induces a trace on  $A_{\theta, \gamma}$  if and only if

$$\int_{\mathbb{T}} f(z) d\mu(e^{2\pi i \theta} z) = \int_{\mathbb{T}} f(z) d\mu(z)$$

for all  $f(z) \in C_0(\widetilde{\mathbb{T} \setminus Y})$ . Suppose that the zero points of  $\gamma(z)$  are  $z_1, \dots, z_n$ . Then the norm closure of  $\gamma(z)C(\mathbb{T})$  in  $C(\mathbb{T})$  is

$$J = \{f(z) \in C(\mathbb{T}) : f(z_1) = \dots = f(z_n) = 0\}$$

and so

$$C = C_0(\widetilde{\mathbb{T} \setminus Y}) = \{f(z) \in C(\mathbb{T}) : f(z_1) = \dots = f(z_n) = 0\} \subseteq C(\mathbb{T}).$$

Therefore,  $\mu$  induces a trace on  $A_{\theta, \gamma}$  if and only if

$$\int_{\mathbb{T}} f(z) d\mu(e^{2\pi i \theta} z) = \int_{\mathbb{T}} f(z) d\mu(z)$$

for all  $f(z) \in C$ .

Let  $C^\perp = \{\rho : \rho \in C(\mathbb{T})^* \text{ and } \rho(a) = 0 \text{ for all } a \in C\}$ . Note that  $C(\mathbb{T})/C \cong \mathbb{C}^{n-1}$ . So  $\dim C^\perp = n - 1$ . Suppose that  $Y_j = \{z_{j,1}, (e^{2\pi i m_{j,1}\theta})z_{j,1}, \dots, (e^{2\pi i m_{j,n_j}\theta})z_{j,1}\}$  with  $1 < m_{j,1} < \dots < m_{j,n_j}$ . Define  $\mu'_{j,k}$ s as in the proof Lemma 3.5 and let  $\nu_j = \delta_{z_{j,1}} - \delta_{z_{j+1,1}}$  for  $1 \leq j \leq r - 1$ . Then

$$\{\mu_{j,k} - \mu_{j,k}(e^{2\pi i \theta} \cdot) : 1 \leq j \leq r, 1 \leq k \leq n_j\} \cup \{\nu_j : 1 \leq j \leq r - 1\}$$

are  $n - 1$  linearly independent elements in  $C^\perp$ . Therefore, there are real numbers  $s_{j,k}$  and  $t_j$  such that

$$\mu(e^{2\pi i \theta} E) - \mu(E) - \sum s_{j,k} (\mu_{j,k}(e^{2\pi i \theta} E) - \mu_{j,k}(E)) = \sum_{j=1}^{r-1} t_j \nu_j(E)$$

for all Borel measurable subset  $E$  of  $\mathbb{T}$ . Let  $\bar{\mu}(E) = \mu(E) - \sum s_{j,k} \mu_{j,k}(E)$ . Then  $\bar{\mu}(\{z_{1,1}\}) = \mu(\{z_{1,1}\}) \geq 0$  and

$$\bar{\mu}(e^{2\pi i \theta} E) - \bar{\mu}(E) = \sum_{j=1}^{r-1} t_j (\delta_{z_{j,1}}(E) - \delta_{z_{j+1,1}}(E)). \quad (3.17)$$

Claim  $t_1 = \dots = t_{r-1} = 0$ . Otherwise, we may assume that  $t_1 > 0$ . In (3.17) let  $E = \{z_{1,1}\}$ , then we have  $\delta_{z_{j,1}}(E) = 0$  for all  $2 \leq j \leq r$ . Hence,  $\bar{\mu}(\{e^{2\pi i \theta} z_{1,1}\}) \geq t_1 + \bar{\mu}(\{z_{1,1}\}) \geq t_1$ . In (3.17) let  $E = \{e^{2\pi i \theta} z_{1,1}\}$ , then we have  $\delta_{z_{j,1}}(E) = 0$  for all  $1 \leq j \leq r$ . Hence,  $\bar{\mu}(\{e^{2\pi i 2\theta} z_{1,1}\}) = \bar{\mu}(\{e^{2\pi i \theta} z_{1,1}\}) \geq t_1 > 0$ . By induction, we have  $\bar{\mu}(\{e^{2\pi i n \theta} z_{1,1}\}) \geq t_1 > 0$  for all  $n \in \mathbb{N}$ . This contradicts to the fact that  $\bar{\mu}$  is a bounded real measure.

Therefore,

$$\mu(e^{2\pi i \theta} E) - \mu(E) = \sum s_{j,k} (\mu_{j,k}(e^{2\pi i \theta} E) - \mu_{j,k}(E))$$

for all Borel measurable subset  $E$  of  $\mathbb{T}$ , i.e.,

$$\mu(e^{2\pi i \theta} E) - \sum s_{j,k} (\mu_{j,k}(e^{2\pi i \theta} E)) = \mu(E) - \sum s_{j,k} (\mu_{j,k}(E))$$

for all Borel subset  $E$  of  $\mathbb{T}$ . Let

$$\nu = \mu - \sum s_{j,k} \mu_{j,k}.$$

Then

$$\nu(e^{2\pi i \theta} E) = \nu(E)$$

for all Borel subsets of  $\mathbb{T}$ . Therefore, for every  $n \in \mathbb{N}$ ,  $\nu(e^{2\pi i n \theta} E) = \nu(E)$  for all Borel subsets  $E$  of  $\mathbb{T}$ . Since  $\theta$  is an irrational number,  $\{e^{2\pi i n \theta} : n \in \mathbb{N}\}$  is dense in  $\mathbb{T}$ . By the Lebesgue dominated theorem,  $\nu(zE) = \nu(E)$  for all Borel subsets  $E$  of  $\mathbb{T}$  and  $z \in \mathbb{T}$ . By the uniqueness of the Haar measure on  $\mathbb{T}$ , there exists  $t \in \mathbb{R}$  such that  $\nu = tm$ , i.e.,  $\mu = \sum s_{j,k} \mu_{j,k} + tm$ . This implies that  $\dim(\text{Tr}(A_{\theta, \gamma})) \leq 1 + \sum_{j=1}^r (|Y_j| - 1)$ . So  $\dim(\text{Tr}(A_{\theta, \gamma})) = 1 + \sum_{j=1}^r (|Y_j| - 1)$ .  $\square$

**Proposition 3.8.** *Suppose  $\gamma(z)$  has finitely many points in its zero set  $Y$  and there are  $\xi_1, \xi_2, \dots, \xi_r \in \Delta$  such that  $Y = \cup_{j=1}^r Y_j$ , where  $Y_j = Y \cap \text{Orb}(\xi_j)$ . Then  $\tau$  and the traces induced by  $\mu'_{j,k}$ s constructed in Lemma 3.5 are precisely the extreme points of  $\text{T}(A_{\theta, \gamma})$ .*

*Proof.* Let  $\sigma$  be a tracial state on  $A_{\theta, \gamma}$  induced by a regular Borel probability measure  $\mu$  on  $\mathbb{T}$ . Then by the proof of Proposition 3.7, there are real numbers  $t, s_{j,k}$  such that

$$\mu(E) = tm(E) + \sum s_{j,k} \mu_{j,k}(E)$$

for all Borel subsets  $E$  of  $\mathbb{T}$ . Since  $m$  and  $\mu'_{j,k}$  are mutually disjoint measures,  $t, s_{j,k} \geq 0$  and  $t + \sum s_{j,k} = 1$ . This shows that  $\tau$  and the traces induced by  $\mu'_{j,k}$ s constructed in Lemma 3.5 are precisely the extreme points of  $\text{T}(A_{\theta, \gamma})$ .  $\square$

**Corollary 3.9.** *Suppose  $\gamma(z)$  has finite zero points. Then  $\tau$  is the unique extreme point in  $\text{T}(A_{\theta, \gamma})$  which is faithful on  $A_{\theta, \gamma}$ .*



## 4 Simplicity of generalized universal $C^*$ -algebras

In this section, we provide a characterization of simplicity of a generalized universal algebra  $A_{\theta, \gamma}$  in terms of the zero points of  $\gamma(z)$ . We begin with the following lemma.

**Lemma 4.1.** *Let  $f_n(z) \in C(\mathbb{T})$  for  $-M \leq n \leq N$ . Then*

$$\|x^k f_k(w)\| \leq \left\| \sum_{n=1}^N x^n f_n(w) + f_0(w) + \sum_{m=1}^M f_{-m}(w) (x^*)^m \right\|,$$

and

$$\|f_{-k}(w)(x^*)^k\| \leq \left\| \sum_{n=1}^N x^n f_n(w) + f_0(w) + \sum_{m=1}^M f_{-m}(w) (x^*)^m \right\|.$$

*Proof.* There is a function  $\gamma_k \in C(\mathbb{T})_+$  such that  $x^k = u^k \gamma_k(w)$ ,  $k = 1, 2, \dots$ , Therefore  $u^{-k} x^k f_k(w) = \gamma_k(w) f_k(w)$ .

Put  $a = \sum_{i=0}^N x^i f_i(w) + \sum_{j=1}^M f_{-j}(w)(x^*)^j$ . Let  $\Phi$  be the conditional expectation. Then

$$\|x^k f_k(w)\| = \|u^{-k} x^k f_k(w)\| = \|\Phi(u^{-k} a)\| \leq \|u^{-k} a\| = \|a\|.$$

So the first part of the lemma follows. The second part follows similarly.  $\square$

**Lemma 4.2.** *Let  $Y_1$  be the set of zero points of functions  $\gamma(e^{2\pi i n \theta} z)$  for  $n \geq 0$ , and let  $Y_2$  be the set of zero points of functions  $\gamma(e^{-2\pi i n \theta} z)$  for  $n \geq 1$ . Then  $A_{\theta, \gamma}$  is a simple algebra if and only if  $Y_1 \cap Y_2 = \emptyset$ .*

*Proof.* Suppose  $Y_1 \cap Y_2 = \emptyset$  and  $J$  is a non-zero ideal of  $A_{\theta, \gamma}$ . Then there is a positive nonzero element  $x$  in  $J$ . Since  $w^j x (w^*)^j \in J$ , the limit formula for  $\Phi(x)$  in Proposition 2.1 shows that  $\Phi(x) \in J \cap C^*(w)$ . Since  $\Phi$  is faithful,  $\Phi(x) > 0$ . So  $J \cap C^*(w)$  is a nontrivial ideal in  $C^*(w)$ , which is contained in a maximal nontrivial ideal

$$I = \{f(w) | f(z) \in C(\mathbb{T}) \text{ and } f(z_0) = 0 \text{ for some } z_0 \in \mathbb{T}\}$$

of  $C^*(w)$ .

Let  $f(z) \in C(\mathbb{T})$  such that  $f(w) \in J \cap C^*(w) \subset I$ . Then  $f(z_0) = 0$ . By (2.7) and (2.8), we have

$$x^* f(w) x = f(e^{2\pi i \theta} w) \gamma(w) \in J \cap C^*(w) \subset I. \quad (4.1)$$

By (2.6) and (2.9), we have

$$x f(w) x^* = f(e^{-2\pi i \theta} w) \gamma(-e^{2\pi i \theta} w) \in J \cap C^*(w) \subset I. \quad (4.2)$$

Case 1. Suppose  $z_0 \in Y_2$ . Then the assumption of the theorem implies that  $z_0 \notin Y_1$ . So (4.1) implies that  $f(e^{2\pi i \theta} z_0) = 0$ . Repeat using (4.1), we have for all  $n \in \mathbb{N}$ ,

$$(x^*)^n f(w) x^n = f(e^{2\pi i n \theta} w) \gamma(e^{2\pi i (n-1) \theta} w) \gamma(e^{2\pi i (n-2) \theta} w) \cdots \gamma(w) \in J \cap C^*(w) \subset I.$$

Thus  $f(e^{2\pi i n \theta} z_0) = 0$  for all  $n \in \mathbb{N}$ . Since  $\{e^{2\pi i n \theta} z_0 : n \in \mathbb{N}\}$  is dense in  $\mathbb{T}$ ,  $f(z) = 0$  for all  $z \in \mathbb{T}$ . This implies that  $J \cap C^*(w)$  is trivial and we obtain a contradiction.

Case 2. Suppose  $z_0 \notin Y_2$ . Then (4.2) implies that  $f(e^{-2\pi i \theta} z_0) = 0$ . Repeat using (4.2), we have for all  $n \in \mathbb{N}$ ,

$$x^n f(w)(x^*)^n = f(e^{-2\pi i n \theta} w) \gamma(e^{-2\pi i n \theta} w) \gamma(e^{-2\pi i (n-1) \theta} w) \cdots \gamma(e^{-2\pi i \theta} w) \in J \cap C^*(w) \subset I.$$

Thus  $f(e^{-2\pi i n \theta} z_0) = 0$  for all  $n \in \mathbb{N}$ . Since  $\{e^{-2\pi i n \theta} z_0 : n \in \mathbb{N}\}$  is dense in  $\mathbb{T}$ ,  $f(z) = 0$  for all  $z \in \mathbb{T}$ . This implies that  $J \cap C^*(w)$  is trivial and we obtain a contradiction.

Conversely, suppose  $Y_1 \cap Y_2 \neq \emptyset$ . We may assume that  $\lambda \in \mathbb{T}$  is a zero point of  $\gamma(e^{2\pi i n \theta} z)$  and  $\gamma(e^{-2\pi i m \theta} z)$ . Consider the subset

$$J = \{f(w) | f(z) \in C(\mathbb{T}) \text{ and } f(e^{2\pi i n \theta} \lambda) = \cdots = f(\lambda) = \cdots = f(e^{-2\pi i m \theta} \lambda) = 0\}$$

of  $C^*(w)$ . Claim that  $I = A_{\theta, \gamma} J A_{\theta, \gamma}$  is a two-sided ideal of  $A_{\theta, \gamma}$ . Otherwise, there exists  $f_i(w) \in J$ ,

$$a_i = \sum_{n=1}^K (x^*)^n g_{-n}^i(w) + g^i(w) + \sum_{n=1}^K g_n^i(w) x^n,$$

and

$$b_i = \sum_{n=1}^K (x^*)^n h_{-n}^i(w) + h^i(w) + \sum_{n=1}^K h_n^i(w) x^n,$$

for sufficiently large  $K \in \mathbb{N}$  such that

$$\left\| \sum_{i=1}^N a_i f_i(w) b_i - 1 \right\| < 1,$$

where  $g_n^i, g^i, h_n^i, h^i \in C(\mathbb{T})$ . By Lemma 4.1 and simple computations, we have

$$\begin{aligned} & \left\| \sum_{i=1}^N g_{-K}^i(e^{2\pi i K \theta} w) f_i(e^{2\pi i K \theta} w) h_K^i(e^{2\pi i K \theta} w) \gamma(e^{2\pi i (K-1) \theta} w) \cdots \gamma(w) + \right. \\ & g_{-(K-1)}^i(e^{2\pi i (K-1) \theta} w) f_i(e^{2\pi i (K-1) \theta} w) h_{K-1}^i(e^{2\pi i (K-1) \theta} w) \gamma(e^{2\pi i (K-2) \theta} w) \cdots \gamma(w) + \cdots + \\ & g_{-1}^i(e^{2\pi i \theta} w) f_i(e^{2\pi i \theta} w) h_1^i(e^{2\pi i \theta} w) \gamma(w) + g^i(w) f_i(w) h^i(w) + g_1^i(w) f_i(e^{-2\pi i \theta} w) h_{-1}^i(w) \gamma(e^{-2\pi i \theta} w) + \cdots + \\ & g_{K-1}^i(w) f_i(e^{-2\pi i (K-1) \theta} w) h_{-(K-1)}^i(w) \gamma(e^{-2\pi i (K-1) \theta} w) \cdots \gamma(e^{-2\pi i \theta} w) + \\ & \left. g_K^i(w) f_i(e^{-2\pi i K \theta} w) h_{-K}^i(w) \gamma(e^{-2\pi i K \theta} w) \cdots \gamma(e^{-2\pi i \theta} w) - 1 \right\| < 1. \end{aligned}$$

Let

$$\bar{f}(z) = \sum_{i=1}^N g_{-K}^i(e^{2\pi i K \theta} z) f_i(e^{2\pi i K \theta} z) h_K^i(e^{2\pi i K \theta} z) \gamma(e^{2\pi i (K-1) \theta} z) \cdots \gamma(z) + \cdots +$$

$$g_{-1}^i(e^{2\pi i\theta}z)f_i(e^{2\pi i\theta}z)h_1^i(e^{2\pi i\theta}z)\gamma(z) + g^i(z)f_i(z)h^i(z) + g_1^i(z)f_i(e^{-2\pi i\theta}z)h_{-1}^i(z)\gamma(e^{-2\pi i\theta}z) + \dots + g_K^i(z)f_i(e^{-2\pi iK\theta}z)h_{-K}^i(z)\gamma(e^{-2\pi iK\theta}z) \dots \gamma(e^{-2\pi i\theta}z).$$

Since  $f_i(z) \in J$ ,  $f_i(e^{2\pi in\theta}\lambda) = \dots = f_i(\lambda) = \dots = f_i(e^{-2\pi im\theta}\lambda) = 0$ . Note that  $\gamma(e^{2\pi in\theta}\lambda) = \gamma(e^{-2\pi im\theta}\lambda) = 0$ . So  $\bar{f}(\lambda) = 0$ . Hence  $\|\bar{f}(z) - 1\| \geq 1$  and  $\|\bar{f}(w) - 1\| \geq 1$ . By Lemma 4.1,

$$\left\| \sum_{i=1}^N a_i f_i(w) b_i - 1 \right\| \geq \|\bar{f}(w) - 1\| \geq 1.$$

This is a contradiction.  $\square$

**Theorem 4.3.** *Let  $\theta$  be an irrational number,  $\gamma \in C(\mathbb{T})$  be a non-negative function, let  $Y$  be the set of zeros of  $\gamma$  and let  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  be the homeomorphism by rotation of  $\theta$ . Then the following are equivalent:*

- (1)  $A_{\theta,\gamma}$  is simple;
- (2)  $A_{\theta,\gamma}$  has a unique tracial state;
- (3)  $\phi^n(Y) \cap Y = \emptyset$  for all integers  $n \neq 0$ .
- (4) For each  $\xi \in \mathbb{T}$ ,  $\text{Orb}(\xi) \cap Y$  contains at most one point.

*Proof.* The equivalence of (2), (3) and (4) follows from Theorem 3.2. Let  $Y_1$  be the set of zero points of functions  $\gamma(e^{2\pi in\theta}z)$  for  $n \geq 0$ , and let  $Y_2$  be the set of zero points of functions  $\gamma(e^{-2\pi in\theta}z)$  for  $n \geq 1$ . By Proposition 2.7, condition (3) is equivalent to  $Y_1 \cap Y_2 = \emptyset$ . By Lemma 4.2, (1) is equivalent to (3).  $\square$

**Corollary 4.4.** *Suppose  $\gamma(z) \in C(\mathbb{T})$  is a positive function with a single zero point. Then  $A_{\theta,\gamma}$  is a simple  $C^*$ -algebra with a unique tracial state.*

**Corollary 4.5.** *Suppose  $\gamma(z) \in C(\mathbb{T})$  is a positive function with two zero points  $z_1, z_2$ . Then  $A_{\theta,\gamma}$  is a simple  $C^*$ -algebra with a unique tracial state if and only if there does not exist integer  $k$  such that  $z_2 = e^{2\pi ik\theta}z_1$ .*

**Corollary 4.6.** *If  $m(\{z | \gamma(z) = 0\}) > 0$ , then  $A_{\theta,\gamma}$  is not simple.*

*Proof.* Let  $Y = \{z | \gamma(z) = 0\}$ . If  $A_{\theta,\gamma}$  is simple, then by Theorem 4.3,  $\phi^n(Y) \cap Y = \emptyset$  for all integers  $n \neq 0$ . Then  $\{\phi^n(Y) : n \in \mathbb{Z}\}$  is a sequence of mutually disjoint subsets. Therefore  $m(Y) = 0$ .  $\square$

## 5 Rieffel's projections in generalized universal algebras

**Lemma 5.1.** *If  $\lambda \in \mathbb{T}$ , then  $A_{\theta, \gamma(z)} \cong A_{\theta, \gamma(\lambda z)}$ .*

*Proof.* Let  $A_{\theta, \gamma(z)} = C^*(x, w)$  and  $A_{\theta, \gamma(\lambda z)} = C^*(x', w')$ . Then  $x', \lambda w'$  satisfy (2.1)-(2.4) for  $\gamma(z)$ . So there is a homomorphism  $\varphi : A_{\theta, \gamma(z)} \rightarrow A_{\theta, \gamma(\lambda z)}$  such that  $\varphi(x) = x'$ ,  $\varphi(w) = \lambda w'$ . By symmetry, there is a homomorphism  $\psi : A_{\theta, \gamma(\lambda z)} \rightarrow A_{\theta, \gamma(z)}$  such that  $\psi(x') = x$ ,  $\psi(w') = \lambda w$ . Hence  $\psi \cdot \varphi(x) = x$  and  $\psi \cdot \varphi(w) = w$ ;  $\varphi \cdot \psi(x') = x'$  and  $\varphi \cdot \psi(w') = w'$ . So  $\varphi$  is an isomorphism from  $A_{\theta, \gamma(z)}$  onto  $A_{\theta, \gamma(\lambda z)}$ .  $\square$

**Lemma 5.2.**  $A_{\theta, \gamma} \cong A_{1-\theta, \gamma}$ .

*Proof.* Let  $A_{\theta, \gamma} = C^*(x, w)$  and  $A_{1-\theta, \gamma} = C^*(x', w')$ . Then  $x', (w')^*$  satisfy (2.1)-(2.4) for  $\theta$  and  $\gamma$ . So there is a homomorphism  $\varphi : A_{\theta, \gamma} \rightarrow A_{1-\theta, \gamma}$  such that  $\varphi(x) = x'$ ,  $\varphi(w) = (w')^*$ . By symmetry, there is a homomorphism  $\psi : A_{1-\theta, \gamma} \rightarrow A_{\theta, \gamma}$  such that  $\psi(x') = x$ ,  $\psi(w') = w^*$ . Hence  $\psi \cdot \varphi(x) = x$  and  $\psi \cdot \varphi(w) = w$ ;  $\varphi \cdot \psi(x') = x'$  and  $\varphi \cdot \psi(w') = w'$ . So  $\varphi$  is an isomorphism from  $A_{\theta, \gamma}$  onto  $A_{1-\theta, \gamma}$ .  $\square$

The proof the following theorem is similar to the proof of Theorem 1.1 of [34]. However, some details should be treated carefully.

**Theorem 5.3.** *Suppose  $\gamma$  is a positive function in  $C(\mathbb{T})$  and there exists  $\lambda \in \mathbb{T}$  such that  $\gamma(\lambda e^{2\pi i n \theta}) \neq 0$  for all nonnegative integers  $n$ . Then for every  $\alpha$  in  $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1]$ , there is a projection  $p$  in  $A_{\theta, \gamma}$  such that  $\tau(p) = \alpha$ .*

*Proof.* By Lemma 5.1, we may assume that  $\lambda = 1$ . Firstly we prove if  $\alpha = \theta \in (0, 1)$  then there exists a projection  $p$  in  $A_{\theta, \gamma}$  such that  $\tau(p) = \theta$ . By Lemma 5.2, we may assume that  $0 < \theta < 1/2$ .

A dense set of elements of  $A_{\theta, \gamma}$  can be represented by a finite sum of the form  $\sum_{i=1}^n f_i(w)x^i + f(w) + \sum_{j=1}^m f_{-j}(w)(x^*)^j$ , where  $f_k(z), f(z) \in C(\mathbb{T})$ . Note that the set  $C(\mathbb{T})x^i, C(\mathbb{T}), C(\mathbb{T})(x^*)^j$  are mutually orthogonal to each other in  $L^2(A_{\theta, \gamma}, \tau)$ . In the following we identify  $C^*(w)$  with  $C(\mathbb{R}/\mathbb{Z})$ . For  $f(t) \in C(\mathbb{R}/\mathbb{Z})$ , define  $f_\theta(t) = f(t - \theta)$ . Let  $\beta(t) = (\gamma(e^{2\pi i t}))^{1/2}$ . Then  $\beta(n\theta) \neq 0$  for all nonnegative integers  $n$ .

We look for a projection  $p = g(t)x + f(t) + h(t)x^*$  such that  $\tau(p) = \theta$ . Since  $p = p^*$ , by (2.4) and (2.5),

$$g(t)x + f(t) + h(t)x^* = x^* \bar{g}(t) + \bar{f}(t) + x \bar{h}(t) = \bar{g}_{-\theta}(t)x^* + \bar{f}(t) + \bar{h}_\theta(t)x.$$

By comparing coefficients, we see that  $f = \bar{f}$  is a real valued function; and that  $h(t) = \overline{g(t + \theta)}$  or equivalently  $h(t - \theta) = \overline{g(t)}$ . Since  $p = p^2$ , (2.6)-(2.9) imply

$$\begin{aligned} g(t)x + f(t) + h(t)x^* &= g(t)g_\theta(t)x^2 + (g(t)(f(t) + f_\theta(t)))x + [g(t)h_\theta(t)\beta^2(t - \theta) + f^2(t) + h(t)g_{-\theta}(t)\beta^2(t)] \\ &\quad + h(t)(f(t) + f_{-\theta}(t))x^* + h(t)h_{-\theta}(t)(x^*)^2. \end{aligned}$$

By comparing coefficients and replacing  $h$ 's with  $g$ 's using the relation between them, we arrive at the necessary and sufficient conditions:

$$g(t)g(t - \theta) = 0, \quad (5.1)$$

$$g(t)(1 - f(t) - f(t - \theta)) = 0, \quad (5.2)$$

$$f(t) - f(t)^2 = |g(t)\beta(t - \theta)|^2 + |g(t + \theta)\beta(t)|^2. \quad (5.3)$$

Pick any positive  $\epsilon > 0$  such that  $\theta + \epsilon < 1/2$ . Define  $f$  to be the piece-wise linear function

$$f(t) = \begin{cases} \epsilon^{-1}t & \text{for } 0 \leq t \leq \epsilon \\ 1 & \text{for } \epsilon \leq t \leq \theta \\ \epsilon^{-1}(\theta + \epsilon - t) & \text{for } \theta \leq t \leq \theta + \epsilon \\ 0 & \text{for } \theta + \epsilon \leq t \leq 1 \end{cases}$$

and define

$$g(t) = \begin{cases} \sqrt{f(t) - f(t)^2}/\beta(t - \theta) & \text{for } \theta \leq t \leq \theta + \epsilon \\ 0 & \text{otherwise} \end{cases}.$$

Since  $\beta(0) \neq 0$ ,  $g(t) \in C(\mathbb{T})$  for sufficiently small  $\epsilon > 0$ . Then  $f(t)$  and  $g(t)$  satisfy equations (5.1), (5.2), and (5.3). So  $\tau(p) = \int_0^1 f(t)dt = \theta$ . We also get the projection  $1 - p$  with trace  $\tau(1 - p) = 1 - \theta$ .

In the following we show that for  $k \geq 2$  there is a projection  $q$  such that  $\tau(q)$  is the fractional part  $k\theta$  of  $k\theta$ . Let  $\alpha = \{k\theta\}$ . We may assume that  $\alpha < 1/2$ . The idea is similar. Let  $q = g_1(t)(u + v)^k + f_1(t) + h_1(t)((u + v)^*)^k$ . Then we will have the following equations

$$g_1(t)g_1(t - \alpha) = 0, \quad (5.4)$$

$$g_1(t)(1 - f_1(t) - f_1(t - \alpha)) = 0, \quad (5.5)$$

$$\begin{aligned} f_1(t) - f_1(t)^2 &= |g_1(t)\beta(t - k\theta) \cdots \beta(t - \theta)|^2 + |g_1(t + \alpha)\beta(t + (k - 1)\theta) \cdots \beta(t)|^2 \\ &= |g_1(t)\beta(t - k\theta)\beta(t - (k - 1)\theta) \cdots \beta(t - \theta)|^2 \\ &\quad + |g_1(t + \alpha)\beta((t + \alpha) - k\theta)\beta((t + \alpha) - (k - 1)\theta) \cdots \beta((t + \alpha) - \theta)|^2. \end{aligned} \quad (5.6)$$

Pick any positive  $\epsilon > 0$  such that  $\theta + \epsilon < 1/2$ . Define  $f_1$  to be the piece-wise linear function

$$f_1(t) = \begin{cases} \epsilon^{-1}t & \text{for } 0 \leq t \leq \epsilon \\ 1 & \text{for } \epsilon \leq t \leq \alpha \\ \epsilon^{-1}(\alpha + \epsilon - t) & \text{for } \alpha \leq t \leq \alpha + \epsilon \\ 0 & \text{for } \alpha + \epsilon \leq t \leq 1 \end{cases}$$

and define

$$g_1(t) = \begin{cases} \sqrt{f_1(t) - f_1(t)^2} / \beta(t - k\theta) \cdots \beta(t - \theta) & \text{for } \alpha \leq t \leq \alpha + \epsilon \\ 0 & \text{otherwise} \end{cases}.$$

Since  $\beta(0) \neq 0$ ,  $\beta(\theta) \neq 0$ ,  $\dots$ ,  $\beta((k-1)\theta) \neq 0$ ,  $g_1(t) \in C(\mathbb{T})$  for sufficiently small  $\epsilon > 0$ . Then  $f_1(t)$  and  $g_1(t)$  satisfy equations (5.4), (5.5), and (5.6). So  $\tau(q) = \int_0^1 f_1(t)dt = \alpha$ .

□

**Corollary 5.4.** *If  $m(\{z | \gamma(z) = 0\}) = 0$ , e.g., the zero points of  $f(z)$  is countable, then for every  $\alpha$  in  $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1]$ , there is a projection  $p$  in  $A_{\theta, \gamma}$  such that  $\tau(p) = \alpha$ .*

*Proof.* We divide  $\mathbb{T}$  into equivalent classes  $F_\alpha$ , where  $x, y \in F_\alpha$  if and only if  $x = e^{2\pi i k \theta} y$  for some  $k \in \mathbb{Z}$ . Suppose  $\forall \alpha$ ,  $F_\alpha \cap \{z | \gamma(z) = 0\} \neq \emptyset$ . By axiom of choice we can choose a representative set  $\{x_\alpha\}_{\alpha \in Y}$  of  $\{F_\alpha\}_{\alpha \in Y}$  such that  $x_\alpha \in F_\alpha \cap \{z | \gamma(z) = 0\}$  for each  $\alpha \in Y$ . Then  $m(\{x_\alpha\}_{\alpha \in Y}) = 0$ . On the other hand it is well-known that  $\{x_\alpha\}_{\alpha \in Y}$  is not Lebesgue measurable. This is a contradiction. Therefore, there exists  $\alpha \in Y$  such that the intersection of  $F_\alpha$  and the set of zero points of  $\gamma$  is empty. Now the corollary follows from Theorem 5.3. □

Combining Corollary 4.6 and Corollary 5.4, we obtain the following result.

**Corollary 5.5.** *If a generalized universal  $C^*$ -algebra  $A_{\theta, \gamma}$  is simple, then for every  $\alpha$  in  $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0, 1]$ , there is a projection  $p$  in  $A_{\theta, \gamma}$  such that  $\tau(p) = \alpha$ .*

This corollary also follows from 7.2.

## 6 K-groups of generalized universal irrational rotation algebras

Let  $A_\theta$  be the universal irrational rotation  $C^*$ -algebra with two unitary generators  $u, v$  satisfying  $vu = e^{2\pi i \theta} uv$ . Then there exists an action  $\alpha_z$  of  $\mathbb{T}$  on  $A_\theta$  defined by  $\alpha_z(u) = zu$  and  $\alpha_z(v) = v$ . By Theorem 2.5, we may identify  $A_{\theta, \gamma}$  with the unital  $C^*$ -subalgebra  $B$  of  $A_\theta$  generated by  $u\gamma^{1/2}(v)$  and  $v$ . Then  $x = u\gamma^{1/2}(v)$  and  $w = v$ . Let  $A$  be the unital  $C^*$ -algebra generated by  $v$ . The following definition is introduced by Ruy Excel in [12].

**Definition 6.1.** For each  $n \in \mathbb{Z}$  the  $n^{th}$  spectral subspace for  $\alpha$  is defined by

$$B_n = \{b \in A_{\theta, \gamma} : \alpha_z(b) = z^n b \text{ for } z \in \mathbb{T}\}.$$

**Lemma 6.2.**  $B_0 = A$  and  $B_1 = \{uf(v) : f(\lambda) = 0 \text{ for } \lambda \in Y\}$ .

*Proof.* By Corollary 2.2,  $B_0 = A$ . We need to show  $B_1 = \{uf(v) : f(\lambda) = 0 \text{ for } \lambda \in Y\}$ . Note that  $\alpha_z(xg(v)) = z xg(v)$ . Since the norm closure of  $\{xg(v) : g \in C(\mathbb{T})\}$  is  $\{uf(v) : f(\lambda) = 0 \text{ for } \lambda \in Y\}$ ,  $\{uf(v) : f(\lambda) = 0 \text{ for } \lambda \in Y\} \subseteq B_1$ . On the other hand, if  $b \in B_1$ , then  $\alpha_z(u^*b) = u^*b$  for all  $z \in \mathbb{T}$ . This implies that  $b = uf(v)$  for some  $f(z) \in C(\mathbb{T})$ . Suppose  $f(\lambda_0) \neq 0$  for some  $\lambda_0 \in Y$ . Then for any  $y = \sum_{n=1}^N x^n f_n(z) + f_0(z) + \sum_{n=1}^N f_{-n}(z)(x^*)^n$ , by Lemma 4.1 we have

$$\|y - z\| \geq \|xf_1(v) - uf(v)\| = \|uh(v)f_1(v) - uf(v)\| = \|h(v)f_1(v) - f(v)\| \geq |f(\lambda_0)| > 0.$$

Thus for any  $y \in A_{\theta, \gamma}$  we have  $\|y - uf(v)\| \geq |f(\lambda_0)| > 0$ . This is a contradiction. So  $B_1 = \{uf(v) : f(\lambda) = 0 \text{ for } \lambda \in Y\}$ .  $\square$

**Definition 6.3.** If  $X$  and  $Y$  are subsets of a  $C^*$ -algebra, then  $XY$  denotes the closed linear span of the set of products  $xy$  with  $x \in X$  and  $y \in Y$ .

**Corollary 6.4.**  $B_1^*B_1 = \{f(v) : f(\lambda) = 0 \text{ for } \lambda \in Y\} \subseteq A$  and  $B_1B_1^* = uB_1B_1^*u^* \subset A$ .

**Lemma 6.5.** The action of  $\mathbb{T}$  on  $A_{\theta, \gamma}$  is regular in the sense of [12] (see Definition 4.4), i.e., there exist an isomorphism  $\theta : B_1^*B_1 \rightarrow B_1B_1^*$  and a linear isometry  $\phi$  from  $B_1^*$  onto  $B_1B_1^*$  such that for  $y_1, y_2 \in B_1$ ,  $a \in B_1^*B_1$  and  $b \in B_1B_1^*$ ,

1.  $\phi(y_1^*b) = \phi(y_1^*)b$ ;
2.  $\phi(ay_1^*) = \theta(a)\phi(y_1^*)$ ;
3.  $\phi(y_1^*)^*\phi(y_2^*) = y_1y_2^*$ ;
4.  $\phi(y_1^*)\phi(y_2^*)^* = \theta(y_1^*y_2)$ .

*Proof.* By Corollary 6.4,  $B_1B_1^* = uB_1B_1^*u^*$ . Let  $\theta(f(v)) = uf(v)u^*$ . Then  $\theta$  is an isomorphism of  $B_1B_1^*$  onto  $B_1^*B_1$ . Define  $\phi(f(v)u^*) = uf(v)u^*$ . By Lemma 6.2 and Corollary 6.4,  $\phi$  is a linear isometry from  $B_1^*$  onto  $B_1B_1^*$ . Let  $y_1 = uf_1(v)$  and  $y_2 = uf_2(v)$  such that  $y_1, y_2 \in B_1$ ,  $a = g_1(v) \in B_1^*B_1 \subset A$  and  $b = g_2(v) \in B_1B_1^* \subset A$ . Then

$$\phi(y_1^*b) = \phi(\bar{f}_1(v)u^*g_2(v)) = \phi(\bar{f}_1(v)\theta^{-1}(g_2(v))u^*) = u\bar{f}_1(v)\theta^{-1}(g_2(v))u^* = u\bar{f}_1(v)u^*g_2(v) = \phi(y_1^*)b,$$

$$\phi(ay_1^*) = \phi(g_1(v)\bar{f}_1(v)u^*) = ug_1(v)\bar{f}_1(v)u^* = \theta(g_1(v))u\bar{f}_1(v)u^* = \theta(a)\phi(y_1^*),$$

$$\phi(y_1^*)^*\phi(y_2^*) = (uy_1^*)^*(uy_2^*) = y_1y_2^*,$$

$$\phi(y_1^*)\phi(y_2^*)^* = (uy_1^*)(uy_2^*)^* = uy_1^*y_2^*u^* = \theta(y_1^*y_2).$$

$\square$

**Lemma 6.6.** *Let  $\Theta = (\theta, B_1^*B_1, B_1B_1^*)$  be the partial automorphism of the fixed point algebra  $A$  as in [12]. Then there exists an isomorphism*

$$\varphi : C^*(A, \Theta) \rightarrow A_{\theta, \gamma}.$$

*Proof.* Clearly  $A_{\theta, \gamma}$  is generated by the fixed point algebra  $A$  and the first spectral subspace  $B_1$ . So the action  $\alpha$  of  $\mathbb{T}$  on  $A_{\theta, \gamma}$  is semi-saturated (see Definition 4.1 of [12]). By Lemma 6.5,  $\alpha$  is also regular. By Theorem 4.21 of [12], there exists an isomorphism

$$\varphi : C^*(A, \Theta) \rightarrow A_{\theta, \gamma}.$$

□

**Theorem 6.7.** *Let  $Y$  be the set of zeros of  $\gamma$ . If  $\mathbb{T} \neq Y \neq \emptyset$ , then*

$$K_1(A_{\theta, \gamma}) = \mathbb{Z} \tag{6.1}$$

*and there exists a splitting short exact sequence:*

$$0 \rightarrow \mathbb{Z} \rightarrow K_0(A_{\theta, \gamma}) \rightarrow C(Y, \mathbb{Z}) \rightarrow 0. \tag{6.2}$$

*In particular, if  $Y$  has  $n$  points, then*

$$K_0(A_{\theta, \gamma}) = \mathbb{Z}^{n+1}. \tag{6.3}$$

*Proof.* Let  $J = B_1B_1^*$ . By Lemma 6.6 and Theorem 7.1 of [12], we have the following exact sequence of  $K$ -groups

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{i_* - \theta_*^{-1}} & K_0(A) & \xrightarrow{i_*} & K_0(A_{\theta, \gamma}) \\ \uparrow & & & & \downarrow \\ K_1(A_{\theta, \gamma}) & \xleftarrow{i_*} & K_1(A) & \xleftarrow{i_* - \theta_*^{-1}} & K_1(J) \end{array}$$

It is easy to see that  $K_0(J) = 0$ ,  $K_1(J) \cong C(Y, \mathbb{Z})$ ,  $K_0(A) \cong K_1(A) \cong \mathbb{Z}$ . Note that

$$K_1(J) \xrightarrow{i_* - \theta_*^{-1}} K_1(A)$$

is the composition of maps

$$K_1(J) \xrightarrow{id} K_1(A)$$

and

$$K_1(A) \xrightarrow{id - \theta_*^{-1}} K_1(A).$$

Since

$$K_1(A) \xrightarrow{id - \theta_*^{-1}} K_1(A)$$



is zero map,

$$K_1(J) \xrightarrow{i_* - \theta_*^{-1}} K_1(A)$$

is zero map. This gives the short exact sequence (6.2).

To see it splits, note that  $Y$  may be identified with a compact subset of the unit line segment which in turn is viewed as a compact subset of the plane. Note also that  $K_1(C(Y)) = \{0\}$ . It follows from the BDF-theory [3] that  $\text{Ext}(C(Y)) = \{0\}$ . Let  $E$  be a unital essential extension of the form:

$$0 \rightarrow \mathcal{K} \rightarrow E \rightarrow C(Y) \rightarrow 0.$$

The fact that  $\text{Ext}(C(Y)) = \{0\}$  implies, in particular, the short exact sequence

$$0 \rightarrow K_0(\mathcal{K}) \rightarrow K_0(E) \rightarrow K_0(C(Y)) \rightarrow 0$$

splits for any such  $E$ , or,

$$0 \rightarrow \mathbb{Z} \rightarrow K_0(E) \rightarrow C(Y, \mathbb{Z}) \rightarrow 0$$

splits for any such group  $K_0(E)$ . It follows (from Brown's UCT) that  $\text{Ext}_{\mathbb{Z}}(\mathbb{Z}, C(Y, \mathbb{Z})) = \{0\}$ . Therefore the short exact sequence (6.2) splits.  $\square$

**Corollary 6.8.**  *$K_i(A_{\theta, \gamma})$  is torsion free,  $i = 0, 1$ . If  $\gamma$  has finitely many zeros, then  $K_i(A_{\theta, \gamma})$  is free,  $i = 0, 1$ .*

## 7 Classification of simple $C^*$ -algebras of $A_{\theta, \gamma}$

In this section, we will discuss the structure of  $A_{\theta, \gamma}$  when it is simple. For recursive subhomogeneous algebras see [28], Section 1. Recall also that the Jiang-Su algebra  $\mathcal{Z}$  is a unital simple  $C^*$ -algebra of recursive subhomogeneous  $C^*$ -algebra with one dimensional base spaces with a unique tracial state and with  $K_0(\mathcal{Z}) = \mathbb{Z}$  and  $K_1(\mathcal{Z}) = \{0\}$  (see [18]).

**Lemma 7.1.** *Let  $\theta$  be an irrational number. Suppose that  $\gamma$  has at least one zero. Then  $A_{\theta, \gamma}$  is an inductive limit of recursive subhomogeneous  $C^*$ -algebras with one dimensional base spaces. In particular, if  $A_{\theta, \gamma}$  is simple, then  $A_{\theta, \gamma}$  is  $\mathcal{Z}$ -stable, where  $\mathcal{Z}$  is the Jiang-Su algebra [18].*

*Proof.* Let  $Y$  be the set of zeros of  $\gamma$ . It is a closed subset of  $\mathbb{T}$ . Let  $1/4 > \epsilon_n > 0$  be such that  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ . Define

$$Y_n = \{x \in \mathbb{T} : \text{dist}(x, Y) \leq \epsilon_n\}, \quad n = 1, 2, \dots$$

Define

$$A_Y = C^*(C(\mathbb{T}), uC_0(\mathbb{T} \setminus Y)) \quad \text{and} \tag{7.1}$$

$$A_{Y_n} = C^*(C(\mathbb{T}), uC_0(\mathbb{T} \setminus Y_n)). \quad (7.2)$$

By Theorem 2.3 of [26] (or Example 1.6 of [28]),  $A_{Y_n}$  is a recursive subhomogeneous  $C^*$ -algebra with one dimensional base spaces. Since  $A_Y = \lim_{n \rightarrow \infty} A_{Y_n}$  (with inclusion maps), the first part of the lemma follows.

To see the second part, it follows from Theorem 1.6 of [38] that each  $A_{Y_n}$  has decomposition rank at most one. Therefore  $A_Y$  has decomposition rank one. Since we assume that  $A_Y$  is simple, by Theorem 5.1 of [39],  $A_Y$  is  $\mathcal{Z}$ -stable. Note that  $A_Y = A_{\theta, \gamma}$ .  $\square$

**Lemma 7.2.** *Suppose that  $A_{\theta, \gamma}$  is simple and  $Y$  is the set of zeros of  $\gamma$ . Let  $\iota$  be the embedding of  $A_{\theta, \gamma} = C^*(C(\mathbb{T}), uC_0(\mathbb{T} \setminus Y)) \subset A_\theta$ , and let  $\rho_{A_{\theta, \gamma}}$  be the induced map of  $K_0(A_{\theta, \gamma})$  into  $K_0(A_\theta)$ . Then*

$$\rho_{A_{\theta, \gamma}}(K_0(A_{\theta, \gamma})) = \mathbb{Z} + \mathbb{Z}\theta \text{ and } \ker \rho_{A_{\theta, \gamma}} \cong C(Y, \mathbb{Z})/\mathbb{Z}, \quad (7.3)$$

where  $\mathbb{Z}$  is identified with constant functions in  $C(Y, \mathbb{Z})$ . Thus one has the following splitting short exact sequence:

$$0 \rightarrow C(Y, \mathbb{Z})/\mathbb{Z} \rightarrow K_0(A_{\theta, \gamma}) \xrightarrow{\rho_{A_{\theta, \gamma}}} \mathbb{Z} + \mathbb{Z}\theta \rightarrow 0. \quad (7.4)$$

Moreover, in this case,

$$K_0(A_{\theta, \gamma})_+ = \{0\} \cup \{x \in K_0(A_{\theta, \gamma}) : \rho_{A_{\theta, \gamma}}(x) > 0\} \quad (7.5)$$

and  $K_0(A_{\theta, \gamma})$  is weakly unperforated and has the Riesz interpolation property.

*Proof.* Denote by  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  the rotation of the unit circle by  $\theta$ , i.e.,  $\phi(z) = e^{2\pi i \theta} z$  for  $z \in \mathbb{T}$ . By the assumption of the lemma and Theorem 4.3,  $\phi^n(Y) \cap Y = \emptyset$  for all integers  $n \neq 0$ . By Theorem 2.4 and Example 2.6 of [33], one obtains the following six term exact sequence:

$$\begin{array}{ccccccc} K_0(C(Y)) & \longrightarrow & K_0(A_{\theta, \gamma}) & \xrightarrow{\iota_{*0}} & K_0(A_\theta) \\ \uparrow & & & & \downarrow \\ K_1(A_\theta) & \xleftarrow{\iota^*1} & K_1(A_{\theta, \gamma}) & \longleftarrow & K_1(C(Y)) \end{array}$$

Note that  $Y$  is a proper closed subset of the circle. It follows that  $K_1(C(Y)) = \{0\}$  and  $\iota_{*0} = \rho_{A_{\theta, \gamma}}$  is surjective. Since  $K_0(A_\theta) = \mathbb{Z} + \mathbb{Z}\theta$  as an ordered subgroup of  $\mathbb{R}$ ,  $\text{Ran} \rho_{A_{\theta, \gamma}} = \mathbb{Z} + \mathbb{Z}\theta$ . By Theorem 6.7,  $K_1(A_{\theta, \gamma}) = \mathbb{Z}$ . One then computes that

$$\ker \rho_{A_{\theta, \gamma}} \cong C(Y, \mathbb{Z})/\mathbb{Z}.$$

It is proved in Lemma 7.1 that  $A_{\theta, \gamma}$  is  $\mathcal{Z}$ -stable. In particular,  $K_0(A_{\theta, \gamma})$  has the strict comparison. Therefore

$$K_0(A_{\theta, \gamma})_+ = \{0\} \cup \{x \in K_0(A_{\theta, \gamma}) : \rho_{A_{\theta, \gamma}}(x) > 0\}. \quad (7.6)$$

It follows that  $K_0(A_{\theta, \gamma})$  is weakly unperforated and has the Riesz interpolation property.  $\square$

For the convenience of the reader, we recall the meaning of tracial rank zero (or tracial topological rank zero) for simple  $C^*$ -algebras.

**Definition 7.3.** Let  $A$  be a simple unital  $C^*$ -algebra. Then  $A$  has tracial rank zero if for every subset  $\mathcal{F} \subset A$ , every  $\epsilon > 0$ , and every nonzero positive element  $c \in A$ , there exists a projection  $p \in A$  and a unital finite dimensional subalgebra  $E \subset pAp$  such that:

- (1)  $\|[a, p]\| < \epsilon$  for all  $a \in \mathcal{F}$ .
- (2)  $\text{dist}(pap, E) < \epsilon$  for all  $a \in \mathcal{F}$ .
- (3)  $1 - p$  is Murray-von Neumann equivalent to a projection in  $\overline{cAc}$ .

This definition is equivalent to the original one following from [23], Proposition 3.8.

**Theorem 7.4.** Let  $A_{\theta, \gamma}$  be a unital simple  $C^*$ -algebra. Then  $A_{\theta, \gamma}$  is a unital simple  $\text{AT}$ -algebra of real rank zero. In particular,  $A_{\theta, \gamma}$  has tracial rank zero.

*Proof.* By Lemma 7.1 and Theorem 7.2,  $A_{\theta, \gamma}$  is  $\mathcal{Z}$ -stable,  $K_0(A_{\theta, \gamma})$  is weakly unperforated and has the Riesz interpolation property. Since  $A_{\theta, \gamma}$  is an inductive limit of type I  $C^*$ -algebras, it satisfies the universal coefficient theorem. By Corollary 6.7,  $K_i(A_{\theta, \gamma})$  is torsion free. Therefore, by [10], there is a unital simple  $\text{AT}$ -algebra  $C$  of real rank zero such that

$$(K_0(C), K_0(C)_+, [1_C], K_1(C)) \cong (K_0(A_{\theta, \gamma}), K_0(A_{\theta, \gamma}), [1_{A_{\theta, \gamma}}], K_1(A_{\theta, \gamma})). \quad (7.7)$$

Let  $U$  be a UHF-algebra of infinite type. Consider  $B = A_{\theta, \gamma} \otimes U$ .  $B$  has a unique tracial state and is approximately divisible. Therefore its projections separate the tracial state space. It follows from [4] that  $B$  has real rank zero. Since  $B$  is  $\mathcal{Z}$ -stable,  $B$  has strict comparison for projections. Therefore  $K_0(B)$  is weakly unperforated. It follows from Lemma 7.1 that  $B$  is a locally type I  $C^*$ -algebra. Then, by applying 5.16 of [24],  $B$  has tracial rank zero. We also note that since  $A_{\theta, \gamma}$  satisfies the universal coefficient theorem, so does  $B$ .

It follows from the classification theorem of [25] (Theorem 5.4) that  $C \otimes \mathcal{Z} \cong A_{\theta, \gamma} \otimes \mathcal{Z}$ . However,  $C$  is  $\mathcal{Z}$ -stable and, by Lemma 7.1,  $A_{\theta, \gamma}$  is also  $\mathcal{Z}$ -stable, one actually has

$$C \cong A_{\theta, \gamma}. \quad (7.8)$$

□

**Corollary 7.5.** *Let  $\theta$  be an irrational number,  $\gamma \in C(\mathbb{T})$  be a non-negative function, let  $Y$  be the set of zeros of  $\gamma$  and let  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  be the homeomorphism by rotation of  $\theta$ . Then the following are equivalent:*

- (1)  $A_{\theta, \gamma}$  is simple;
- (2)  $A_{\theta, \gamma}$  has a unique tracial state;
- (3)  $\phi^n(Y) \cap Y = \emptyset$  for all integers  $n \neq 0$ ;
- (4) For each  $\xi \in \mathbb{T}$ ,  $\text{Orb}(\xi) \cap Y$  contains at most one point;
- (5)  $A_{\theta, \gamma}$  is a unital simple  $AT$ -algebra of real rank zero.

**Theorem 7.6.** *Let  $\theta_1$  and  $\theta_2$  be two irrational numbers,  $\gamma_1$  and  $\gamma_2 \in C(\mathbb{T})$  be non-negative functions and let  $Y_i$  be the set of zeros of  $\gamma_i$ ,  $i = 1, 2$ . Suppose that  $A_{\theta_i, \gamma_i}$  is simple, or one of the equivalent conditions in Corollary 7.5 satisfies. Then  $A_{\theta_1, \gamma_1} \cong A_{\theta_2, \gamma_2}$  if and only if the following hold:*

$$\theta_1 = \pm \theta_2 \text{ mod } (\mathbb{Z}) \text{ and } C(Y_1, \mathbb{Z})/\mathbb{Z} \cong C(Y_2, \mathbb{Z})/\mathbb{Z}. \quad (7.9)$$

*In particular, when  $\gamma_1$  has only finitely many zeros, then  $A_{\theta_1, \gamma_1} \cong A_{\theta_2, \gamma_2}$  if and only if  $\theta_1 = \pm \theta_2 \text{ mod } \mathbb{Z}$  and  $\gamma_2$  has the same number of zeros.*

*Proof.* We will prove the “if” part only. Note that we have  $K_1(A_{\theta_1, \gamma_1}) \cong K_1(A_{\theta_2, \gamma_2})$ . We may write, by Lemma 7.2, that

$$K_0(A_{\theta_i, \gamma_i}) = C(Y_i, \mathbb{Z})/\mathbb{Z} \oplus (\mathbb{Z} + \mathbb{Z}\theta). \quad (7.10)$$

It follows that  $K_0(A_{\theta_1, \gamma_1}) \cong K_0(A_{\theta_2, \gamma_2})$ . In fact they are order isomorphic. By Theorem 7.4 both  $C^*$ -algebras are unital simple  $AT$ -algebras of real rank zero. By the classification theorem they are isomorphic. □

**Corollary 7.7.** *With the same assumption as in 7.6, if  $Y_1$  and  $Y_2$  are homeomorphic and  $\theta_1 = \pm \theta_2 \text{ mod } \mathbb{Z}$ , then  $A_{\theta_1, \gamma_1} \cong A_{\theta_2, \gamma_2}$ .*

**Theorem 7.8.** *Let  $\theta_1, \theta_2 \in (0, 1)$  be two irrational numbers,  $\gamma_1, \gamma_2 \in C(\mathbb{T})$  be non-negative functions and let  $Y_i$  be the set of zeros of  $\gamma_i$ ,  $i = 0, 1$ . Suppose that  $A_{\theta_i, \gamma_i}$  is simple, or one of the equivalent conditions in Corollary 7.5 satisfies. Then  $A_{\theta_1, \gamma_1}$  and  $A_{\theta_2, \gamma_2}$  are Morita equivalent if and only if  $\mathbb{Z} + \mathbb{Z}\theta_1$  and  $\mathbb{Z} + \mathbb{Z}\theta_2$  are order isomorphic and*

$$C(Y_1, \mathbb{Z})/\mathbb{Z} \cong C(Y_2, \mathbb{Z})/\mathbb{Z}. \quad (7.11)$$

*In particular, assuming, in addition,  $Y_1$  and  $Y_2$  are both finite subsets, then  $A_{\theta_1, \gamma_1}$  and  $A_{\theta_2, \gamma_2}$  are Morita equivalent if and only if  $\mathbb{Z} + \mathbb{Z}\theta_1$  and  $\mathbb{Z} + \mathbb{Z}\theta_2$  are order isomorphic and  $Y_1$  and  $Y_2$  have the same number of points.*

*Proof.* Suppose that  $h_1 : \mathbb{Z} + \mathbb{Z}\theta_1 \rightarrow \mathbb{Z} + \mathbb{Z}\theta_2$  is an order isomorphism and  $h_2 : C(Y_1, \mathbb{Z})/\mathbb{Z} \rightarrow C(Y_2, \mathbb{Z})/\mathbb{Z}$  is an isomorphism as groups. There is an injective homomorphism  $\iota_i : \mathbb{Z} + \mathbb{Z}\theta_i \rightarrow K_0(A_{\theta_i, \gamma_i})$  such that

$$\rho_{A_{\theta_i, \gamma_i}} \circ \iota_i = \text{id}_{\mathbb{Z} + \mathbb{Z}\theta_i}, \quad i = 1, 2.$$

We write

$$K_0(A_{\theta_i, \gamma_i}) = C(Y_i, \mathbb{Z})/\mathbb{Z} \oplus \iota_i(\mathbb{Z} + \mathbb{Z}\theta_i),$$

$i = 1, 2$ .

Define  $h_3 : K_0(A_{\theta_1, \gamma_1}) \rightarrow K_0(A_{\theta_2, \gamma_2})$  by

$$h_3|_{\ker \rho_{A_{\theta_1, \gamma_1}}} = h_2 \tag{7.12}$$

and

$$h_3(x) = \iota_2 \circ h_1 \circ \rho_{A_{\theta_1, \gamma_1}}(x). \tag{7.13}$$

for  $x \in \iota_1(\mathbb{Z} + \mathbb{Z}\theta_1)$ . It is easy to verify that  $h_3$  is an order isomorphism from  $K_0(A_{\theta_1, \gamma_1})$  onto  $K_0(A_{\theta_2, \gamma_2})$ . We also have  $K_1(A_{\theta_1, \gamma_1}) = \mathbb{Z} = K_1(A_{\theta_2, \gamma_2})$ . Since both  $A_{\theta_1, \gamma_1}$  and  $A_{\theta_2, \gamma_2}$  are unital simple AT-algebras of real rank zero, by the classification results mentioned earlier,  $A_{\theta_1, \gamma_1}$  and  $A_{\theta_2, \gamma_2}$  are stably isomorphic. In other words,  $A_{\theta_1, \gamma_1}$  and  $A_{\theta_2, \gamma_2}$  are Morita equivalent.

Conversely, if  $A_{\theta_1, \gamma_1} \otimes \mathcal{K} \cong A_{\theta_2, \gamma_2} \otimes \mathcal{K}$ , then  $K_0(A_{\theta_1, \gamma_1})$  and  $K_0(A_{\theta_2, \gamma_2})$  are order isomorphic. Denote by  $h_0$  the order isomorphism. This implies, in particular,  $h_0$  maps  $\ker \rho_{A_{\theta_1, \gamma_1}}$  isomorphically onto  $\ker \rho_{A_{\theta_2, \gamma_2}}$  which implies that

$$C(Y_1, \mathbb{Z})/\mathbb{Z} = \ker \rho_{A_{\theta_1, \gamma_1}} \cong \ker \rho_{A_{\theta_2, \gamma_2}} = C(Y_2, \mathbb{Z})/\mathbb{Z}.$$

Therefore  $h_0$  induces an order isomorphism from  $\rho_{A_{\theta_1, \gamma_1}}(K_0(A_{\theta_1, \gamma_1}))$  onto  $\rho_{A_{\theta_2, \gamma_2}}(K_0(A_{\theta_2, \gamma_2}))$  which implies that  $\mathbb{Z} + \mathbb{Z}\theta_1$  and  $\mathbb{Z} + \mathbb{Z}\theta_2$  are order isomorphic.  $\square$

Let  $\text{GL}(2, \mathbb{Z})$  denote the group of  $2 \times 2$  matrices with entries in  $\mathbb{Z}$  and with determinant  $\pm 1$ , and let  $\text{GL}(2, \mathbb{Z})$  act on the set of irrational numbers by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha = \frac{a\alpha + b}{c\alpha + d}.$$

By Corollary 2.6 of [34] (or Lemma 4.7 of [35]),  $\mathbb{Z} + \mathbb{Z}\theta_1$  and  $\mathbb{Z} + \mathbb{Z}\theta_2$  are ordered isomorphic if and only if  $\theta_1$  and  $\theta_2$  are in the same orbit of  $\text{GL}(2, \mathbb{Z})$ . Thus we obtain the following corollary.

**Corollary 7.9.** *Let  $\theta_1, \theta_2 \in (0, 1)$  be two irrational numbers,  $\gamma \in C(\mathbb{T})$  be non-negative functions and let  $Y$  be the set of zeros of  $\gamma$ . Suppose that  $A_{\theta_i, \gamma}$  is simple, or one of the equivalent conditions in Corollary 7.5 satisfies. Then  $A_{\theta_1, \gamma}$  and  $A_{\theta_2, \gamma}$  are morita equivalent if and only if  $\theta_1$  and  $\theta_2$  are in the same orbit under the action of  $\text{GL}(2, \mathbb{Z})$  on irrational numbers.*

## 8 The $C^*$ -algebra generated by $u + \lambda v$

**Proposition 8.1.** *Let  $R$  be the hyperfinite type  $\text{II}_1$  factor with two unitary generators  $u, v$  such that  $vu = e^{2\pi i\theta}uv$ . If  $f(z) \in C(\mathbb{T})$  and  $m(\{z | f(z) = 0\}) = 0$ , then the von Neumann subalgebra generated by  $uf(v)$  and  $v$  is  $R$ . Furthermore,  $C^*(uf(v), v) = C^*(u, v)$  if and only if  $f(z) \neq 0$  for all  $z \in \mathbb{T}$ .*

*Proof.* Let  $M$  be the von Neumann algebra generated by  $uf(v)$  and  $v$ . Since  $m(\{z | f(z) = 0\}) = 0$ ,  $f(v)^{-1}$  is affiliated with  $M$ , i.e., the spectral projections of the unbounded operator  $f(v)^{-1}$  are in  $M$ . Hence  $u = uf(v) \cdot f(v)^{-1}$  is affiliated with  $M$ . Since  $u$  is a bounded operator,  $u \in M$  and therefore  $R \subseteq M$  and  $M = R$ .

If  $f(z) \neq 0$  for all  $z \in \mathbb{T}$ , then  $f(v)$  is an invertible operator in  $C^*(v)$ . Hence  $u = uf(v) \cdot f(v)^{-1}$  is in the  $C^*$ -subalgebra generated by  $uf(v)$  and  $v$ . Therefore,  $C^*(uf(v), v) = C^*(u, v)$ . Conversely, suppose  $f(z_0) = 0$  for some  $z_0 \in \mathbb{T}$ . By Theorem 6.7,  $K_1(C^*(uf(v), v)) \cong \mathbb{Z}$ . Therefore,  $C^*(uf(v), v) \neq C^*(u, v)$ .  $\square$

**Theorem 8.2.** *Let  $R$  be the hyperfinite type  $\text{II}_1$  factor with two unitary generators  $u, v$  such that  $vu = e^{2\pi i\theta}uv$ . Then the von Neumann subalgebra generated by  $u + \lambda v$  is  $R$  for  $\lambda > 0$ . Furthermore,  $C^*(u + \lambda v) = C^*(u, v)$  if  $\lambda \neq 1$  while  $C^*(u + v)$  is a proper simple  $C^*$ -subalgebra of  $C^*(u, v)$  which has a unique trace,  $K_1(C^*(u + v)) \cong \mathbb{Z}$ , and there is an order isomorphism of  $K_0(C^*(u + v))$  onto  $\mathbb{Z} + \mathbb{Z}\theta$ . Moreover,  $C^*(u + v)$  is a unital simple  $\text{AT}$ -algebra of tracial rank zero.*

*Proof.* Note that

$$(u + \lambda v)(u + \lambda v)^* = (u + \lambda v)(u^* + \lambda v^*) = \lambda e^{-2\pi i\theta} u^* v + \lambda u v^* + 1 + \lambda^2$$

and

$$(u + \lambda v)^*(u + \lambda v) = (u^* + \lambda v^*)(u + \lambda v) = u^* v + e^{-2\pi i\theta} \lambda u v^* + 1 + \lambda^2.$$

Hence  $u^* v, uv^* \in C^*(u + \lambda v)$ . Let  $w = u^* v$ . Thus  $C^*(u + \lambda v) = C^*(u + \lambda v, w) = C^*(u(1 + \lambda w), w)$ . By Proposition 8.1, the von Neumann subalgebra generated by  $u + \lambda v$  is  $R$  for  $\lambda > 0$ , and  $C^*(u + \lambda v) = C^*(u, v)$  if  $\lambda \neq 1$  while  $C^*(u + v)$  is a proper  $C^*$ -subalgebra of  $C^*(u, v)$ . Note that  $u + v$  and  $w$  satisfy (2.1)-(2.4) for  $\theta$  and  $\gamma(z) = |1 + z|^2$ . By Proposition 3.2, Theorem 4.2, Theorem 5.3, and Theorem 6.7,  $C^*(u + v)$  is a simple algebra with a unique trace,  $K_1(C^*(u + v)) \cong \mathbb{Z}$ , and there is an order isomorphism of  $K_0(C^*(u + v))$  onto  $\mathbb{Z} + \mathbb{Z}\theta$ . By Theorem 7.4,  $C^*(u + v)$  is a unital simple  $\text{AT}$ -algebra of tracial rank zero.  $\square$

**Corollary 8.3.**  *$C^*(u + v)$  is not  $*$ -isomorphic to  $C^*(u, v)$ .*

## 9 Spectral radius of $u + \lambda v$

In this section, we assume that  $0 \leq \lambda \leq 1$ . Let  $\alpha = e^{2\pi i \theta}$  and  $w = u^*v$ . Then  $w$  is a Haar unitary operator in  $R$ , i.e.,  $\tau(w^n) = \tau((w^*)^n) = 0$  for all  $n \in \mathbb{N}$ . Note that

$$u + \lambda v = u(1 + \lambda u^*v) = u(1 + \lambda w),$$

$$\begin{aligned} (u + \lambda v)^2 &= (u + \lambda v)u(1 + \lambda w) = (u^2 + \alpha \lambda uv)(1 + \lambda w) = u^2(1 + \alpha \lambda u^*v)(1 + \lambda w) = u^2(1 + \alpha \lambda w)(1 + \lambda w), \\ (u + \lambda v)^3 &= (u + \lambda v)u^2(1 + \alpha \lambda w)(1 + \lambda w) = (u^3 + \alpha \lambda u^2v)(1 + \alpha \lambda w)(1 + \lambda w) = u^3(1 + \alpha^2 \lambda w)(1 + \alpha \lambda w)(1 + \lambda w). \end{aligned}$$

By induction, we have

$$(u + \lambda v)^n = u^n(1 + \lambda w)(1 + \alpha \lambda w) \cdots (1 + \alpha^{(n-1)} \lambda w), \quad \forall n \in \mathbb{N} \quad (9.1)$$

Let  $r(u + \lambda v)$  be the spectral radius of  $u + \lambda v$ . Then

$$r(u + \lambda v) = \lim_{n \rightarrow +\infty} \|(u + \lambda v)^n\|^{1/n} = \lim_{n \rightarrow +\infty} \|(1 + \lambda w)(1 + \alpha \lambda w) \cdots (1 + \alpha^{(n-1)} \lambda w)\|^{1/n}.$$

Since  $w = u^*v$  is a Haar unitary operator, we may identify  $w$  with the multiplication operator  $M_z$  on  $L^2(\mathbb{T}, m)$ , where  $m$  is the Haar measure on  $\mathbb{T}$ . Hence,

$$\begin{aligned} \|(u + \lambda v)^n\|^{1/n} &= \|(1 + \lambda w)(1 + \alpha \lambda w) \cdots (1 + \alpha^{(n-1)} \lambda w)\|^{1/n} \\ &= \|(1 + \lambda M_z)(1 + \alpha \lambda M_z) \cdots (1 + \alpha^{(n-1)} \lambda M_z)\|^{1/n} \\ &= \left( \max_{z \in \mathbb{T}} |(1 + \lambda z)(1 + \alpha \lambda z) \cdots (1 + \alpha^{(n-1)} \lambda z)| \right)^{1/n}. \end{aligned}$$

Let  $z = e^{i2\pi x}$ ,  $x \in [0, 1]$ . Then simple calculation shows that

$$|(1 + \lambda z)(1 + \alpha \lambda z) \cdots (1 + \alpha^{(n-1)} \lambda z)| = \left( \prod_{k=0}^{n-1} (1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))) \right)^{\frac{1}{2}}.$$

So

$$\|(u + \lambda v)^n\|^{1/n} = \max_{x \in [0, 1]} \left( \prod_{k=0}^{n-1} (1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))) \right)^{\frac{1}{2n}}. \quad (9.2)$$

**Lemma 9.1.** For  $0 \leq \lambda \leq 1$ ,

$$\int_0^1 \ln(1 + \lambda^2 + 2\lambda \cos 2\pi x) dx = 0.$$

*Proof.* For  $0 \leq \lambda \leq 1$ , let

$$f(\lambda) = \int_0^1 \ln(1 + \lambda^2 + 2\lambda \cos 2\pi x) dx.$$

Then  $f(\lambda)$  is continuous on  $[0, 1]$ , differentiable in  $(0, 1)$ , and  $f(0) = 0$ . Note that for  $0 < \lambda < 1$ ,

$$\begin{aligned}
f'(\lambda) &= \int_0^1 \frac{2\lambda + 2\cos 2\pi x}{1 + \lambda^2 + 2\lambda \cos 2\pi x} dx \\
&= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{2\lambda + z + \frac{1}{z}}{1 + \lambda^2 + \lambda(z + \frac{1}{z})} \cdot \frac{dz}{z} \\
&= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{2\lambda z + z^2 + 1}{(1 + \lambda^2)z + \lambda z^2 + \lambda} \cdot \frac{dz}{z} \\
&= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{2\lambda z + z^2 + 1}{(\lambda z + 1)(z + \lambda)z} dz \\
&= \text{Res} \left( \frac{2\lambda z + z^2 + 1}{(\lambda z + 1)(z + \lambda)z}; 0 \right) + \text{Res} \left( \frac{2\lambda z + z^2 + 1}{(\lambda z + 1)(z + \lambda)z}; -\lambda \right) \\
&= \frac{1}{\lambda} - \frac{1}{\lambda} = 0.
\end{aligned}$$

So for  $0 \leq \lambda \leq 1$ ,  $f(\lambda) = 0$ .

□

**Lemma 9.2.** *Let  $0 < \lambda \leq 1$ . Then for almost all  $x \in [0, 1]$ ,*

$$\lim_{n \rightarrow +\infty} \left( \prod_{k=0}^{n-1} (1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))) \right)^{\frac{1}{2n}} = 1.$$

*Proof.* We only need to show that for almost all  $x \in [0, 1]$ ,

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln(1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))) = 0.$$

Let  $f(x) = \ln(1 + \lambda^2 + 2\lambda \cos 2\pi x)$ . If  $0 < \lambda < 1$ , then

$$2\ln(1 - \lambda) \leq f(x) \leq 2\ln(1 + \lambda), \quad \forall x \in [0, 1].$$

So  $f(x) \in L^1[0, 1]$ . If  $\lambda = 1$ , then

$$f(x) = \ln(2 + 2\cos 2\pi x) = 2\ln 2 + 2\ln |\cos \pi x|$$

and so

$$|f(x)| \leq 2\ln 2 - 2\ln |\cos \pi x|, \quad \forall x \in [0, 1].$$

By Lemma 9.1,  $\int_0^1 f(x) dx = 0$ , which implies that  $\int_0^1 2\ln |\cos \pi x| dx = -2\ln 2$ . Therefore,  $\int_0^1 |f(x)| dx \leq 4\ln 2$  and  $f(x) \in L^1[0, 1]$ .

Let  $T : x \rightarrow x + \theta \pmod{1}$ . Then  $T$  is a measure preserving ergodic transformation of  $[0, 1]$ . By Birkhoff's Ergodic theorem and Lemma 9.1, for almost all  $x \in [0, 1]$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln(1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))) = \int_0^1 \ln(1 + \lambda^2 + 2\lambda \cos 2\pi x) dx = 0.$$

□



**Corollary 9.3.** For  $0 < \lambda \leq 1$ ,  $r(u + \lambda v) \geq 1$ .

*Proof.* Let  $\epsilon > 0$ . By Lemma 9.2, there is an  $x \in [0, 1]$  and  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$\left( \prod_{k=0}^{n-1} (1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))) \right)^{\frac{1}{2n}} \geq 1 - \epsilon.$$

By equation (9.2), for  $n \geq N$ ,

$$\|(u + \lambda v)^n\|^{1/n} \geq 1 - \epsilon.$$

This implies that

$$r(u + \lambda v) = \lim_{n \rightarrow +\infty} \|(u + \lambda v)^n\|^{1/n} \geq 1 - \epsilon.$$

Since  $\epsilon > 0$  is arbitrary,  $r(u + \lambda v) \geq 1$ . □

Let  $\theta \in (0, 1)$  be an irrational number and let  $\alpha = e^{2\pi i \theta}$ .

**Lemma 9.4.** Given  $\epsilon > 0$  and  $N \in \mathbb{N}$ . Then there exists  $N' \in \mathbb{N}$  such that for  $n \geq N'$  and every arc  $\Gamma$  of the unit circle  $\mathbb{T}$  with length  $\frac{2\pi}{N}$ , there exists  $\frac{n}{N} + r$  points of  $1, \alpha, \dots, \alpha^{n-1}$  in  $\Gamma$  with  $|\frac{r}{n}| < \epsilon$ .

*Proof.* Since  $\theta \in (0, 1)$  is irrational,  $\{\alpha^k : k \in \mathbb{N}\}$  is dense in the unit circle  $\mathbb{T}$ . Therefore, there exists  $m \in \mathbb{N}$  such that for every  $0 \leq \varphi \leq 2\pi$ , there exists  $1 \leq k \leq m$  such that  $|(\varphi - 2k\pi\theta) \bmod 2\pi| < \frac{\epsilon}{8}$ . By Birkhoff's ergodic theorem, there exists an arc  $\Gamma_1$  of the unit circle with length  $l(\Gamma_1) = 2\pi(\frac{1}{N} - \frac{\epsilon}{4})$  and

$$\lim_{n \rightarrow \infty} \frac{\chi_{\Gamma_1}(1) + \chi_{\Gamma_1}(\alpha) + \dots + \chi_{\Gamma_1}(\alpha^{n-1})}{n} = \frac{l(\Gamma_1)}{2\pi} = \frac{1}{N} - \frac{\epsilon}{4}.$$

Let  $N_1$  be sufficiently large such that  $\frac{m}{N_1} < \frac{\epsilon}{2}$  and if  $n \geq N_1$  then

$$\frac{\chi_{\Gamma_1}(1) + \chi_{\Gamma_1}(\alpha) + \dots + \chi_{\Gamma_1}(\alpha^{n-1})}{n} \geq \frac{1}{N} - \frac{\epsilon}{4} - \frac{\epsilon}{4} = \frac{1}{N} - \frac{\epsilon}{2}.$$

Let  $e^{2\pi i \theta}$  and  $e^{2\pi i(\theta + 2\pi/N)}$  be the ending points of the arc  $\Gamma$ . Let  $\Gamma'_1 \subset \Gamma$  be the arc of  $\mathbb{T}$  with ending points  $e^{2\pi i \theta + (\pi/4)\epsilon i}$  and  $e^{2\pi i(\theta + 2\pi/N) - (\pi/4)\epsilon i}$ . Then there exists an  $\varphi$  with  $0 \leq \varphi \leq 2\pi$  such that we can rotate  $\Gamma_1$  by angle  $\varphi$  to obtain  $\Gamma'_1$ . So if  $\{\alpha^{k_1}, \dots, \alpha^{k_s}\} \subseteq \Gamma_1$  with  $0 \leq k_1 < k_2 < \dots < k_s \leq n-1$ , then  $\{\alpha^{k_1} e^{i\varphi}, \dots, \alpha^{k_s} e^{i\varphi}\} \subseteq \Gamma'_1 \subseteq \Gamma$ . Since  $|(\varphi - 2k\pi\theta) \bmod 2\pi| < \frac{\epsilon}{8}$  for some  $1 \leq k \leq m$ ,

$$\{\alpha^{k_1} e^{2k\pi\theta i}, \dots, \alpha^{k_{s-m}} e^{2k\pi\theta i}\} \subset \Gamma.$$

Since  $k_{s-m} \leq n - m$ ,  $\{\alpha^{k_1+k}, \dots, \alpha^{k_{s-m}+k}\} \subset \Gamma$ . So  $\Gamma$  contains at least  $n(\frac{1}{N} - \frac{\epsilon}{2}) - m = n(\frac{1}{N} - \epsilon)$  points of  $1, \alpha, \dots, \alpha^{n-1}$ .

By Birkhoff's ergodic theorem, there exists an arc  $\Gamma_2$  of the unit circle with length  $l(\Gamma_2) = 2\pi(\frac{1}{N} + \frac{\epsilon}{4})$  and

$$\lim_{n \rightarrow \infty} \frac{\chi_{\Gamma_2}(1) + \chi_{\Gamma_2}(\alpha) + \dots + \chi_{\Gamma_2}(\alpha^{n-1})}{n} = \frac{l(\Gamma_2)}{2\pi} = \frac{1}{N} + \frac{\epsilon}{4}.$$

Let  $N_2$  be sufficiently large such that  $\frac{m}{N_2} < \frac{\epsilon}{2}$  and if  $n \geq N_2$  then

$$\frac{\chi_{\Gamma_2}(1) + \chi_{\Gamma_2}(\alpha) + \cdots + \chi_{\Gamma_2}(\alpha^{n-1})}{n} \leq \frac{1}{N} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{1}{N} + \frac{\epsilon}{2}.$$

Let  $e^{2\pi i\theta'}$  and  $e^{2\pi i\theta' + 2\pi(1/N + \epsilon/4)i}$  be the ending points of the arc  $\Gamma_2$ . Let  $\Gamma'_2 \subset \Gamma_2$  be the arc of  $\mathbb{T}$  with ending points  $e^{2\pi i\theta' + (\pi/4)\epsilon i}$  and  $e^{2\pi i(\theta' + 1/N) + (\pi/4)\epsilon i}$ . Then there exists an  $\varphi'$  with  $0 \leq \varphi' \leq 2\pi$  such that we can rotate  $\Gamma$  by angle  $\varphi'$  to obtain  $\Gamma'_2$ . So if  $\{\alpha^{j_1}, \dots, \alpha^{j_r}\} \subseteq \Gamma$  with  $0 \leq j_1 < j_2 < \cdots < j_s \leq n-1$ , then  $\{\alpha^{j_1}e^{i\varphi'}, \dots, \alpha^{j_r}e^{i\varphi'}\} \subseteq \Gamma'_2 \subseteq \Gamma$ . Since  $|(\varphi' - 2k'\pi\theta) \bmod 2\pi| < \frac{\epsilon}{8}$  for some  $1 \leq k' \leq m$ ,

$$\{\alpha^{j_1}e^{2k'\pi\theta i}, \dots, \alpha^{j_r-m}e^{2k'\pi\theta i}\} \subset \Gamma_2.$$

So  $\Gamma$  contains at most  $n\left(\frac{1}{N} + \frac{\epsilon}{2}\right) + m = n\left(\frac{1}{N} + \epsilon\right)$  points of  $1, \alpha, \dots, \alpha^{n-1}$ . Let  $N' = \max\{N_1, N_2\}$ . Then we obtain the lemma.  $\square$

Now we prove the main result of this section.

**Lemma 9.5.** For  $0 < \lambda \leq 1$ ,  $r(u + \lambda v) = 1$ .

*Proof.* By Corollary 9.3, we need to prove that  $r(u + \lambda v) \leq 1$ . Let  $\epsilon > 0$ . Note that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{2n} \left( \sum_{k=0}^{n-1} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{n} \right) \right) + \sum_{k=n+1}^{2n} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{n} \right) \right) \right) \\ = \int_0^1 \ln(1 + \lambda^2 + 2\lambda \cos 2\pi x) dx = 0. \end{aligned}$$

There is  $N \in \mathbb{N}$  such that

$$\frac{1}{2N} \left( \sum_{k=0}^{N-1} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) + \sum_{k=N+1}^{2N} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \right) < \epsilon.$$

Let

$$M(\lambda) = \max_{0 \leq k \leq 2N, k \neq N} \left| \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \right|.$$

Then for  $0 < \lambda \leq 1$ ,  $M(\lambda) < \infty$ . Divide the unit circle  $\mathbb{T}$  into  $2N$  equal sections  $A_1, \dots, A_{2N}$ . By Lemma 9.4, there exists  $N'$  such that for all  $n \geq N'$  and all  $x \in [0, 1]$ , if  $A_k$  contains  $n/(2N) + r_k(x)$  points of  $e^{2\pi i x}, \alpha e^{2\pi i x}, \dots, \alpha^{n-1} e^{2\pi i x}$ , then  $\frac{\sum_{k=1}^{2N} |r_k(x)|}{n} < \frac{\epsilon}{M(\lambda)}$ . Note that  $\cos 2\pi x$  is decreasing for  $x \in [0, 1/2]$  and increasing for  $x \in [1/2, 1]$ . Therefore, for all  $x \in [0, 1]$ ,

$$\frac{1}{n} \sum_{k=0}^{n-1} \ln(1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))) \leq \frac{1}{n} \sum_{k=0}^{N-1} \left( \frac{n}{2N} + r_{k+1}(x) \right) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right)$$

$$\begin{aligned}
& + \frac{1}{n} \sum_{k=N+1}^{2N} \left( \frac{n}{2N} + r_k(x) \right) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \\
& = \frac{1}{2N} \left( \sum_{k=0}^{N-1} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) + \sum_{k=N+1}^{2N} \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \right) \\
& + \frac{1}{n} \sum_{k=0}^{N-1} r_{k+1}(x) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) + \frac{1}{n} \sum_{k=N+1}^{2N} r_k(x) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \\
& < \epsilon + \frac{1}{n} \sum_{k=1}^{2N} |r_k(x)| M(\lambda) < 2\epsilon.
\end{aligned}$$

This implies that for all  $n \geq N'$  and  $x \in [0, 1]$ ,

$$\left( \prod_{k=0}^{n-1} (1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))) \right)^{\frac{1}{2n}} \leq e^{2\epsilon}.$$

By equation(9.2),  $\|(u + \lambda v)^n\|^{1/n} \leq e^{2\epsilon}$  for all  $n \geq N'$ . So  $r(u + \lambda v) \leq e^{2\epsilon}$ . Since  $\epsilon > 0$  is arbitrary,  $r(u + \lambda v) \leq 1$ .  $\square$

## 10 Strongly irreducible operators relative to type II<sub>1</sub> factors

An operator  $T$  in a type II<sub>1</sub> factor  $M$  is called *irreducible* if  $\{T, T^*\}' \cap M = \mathbb{C}1$ , i.e., the von Neumann subalgebra generated by  $T$  is an irreducible subfactor of  $M$ .

**Proposition 10.1.** *Every separable type II<sub>1</sub> factor  $M$  contains an irreducible operator.*

*Proof.* By [29], every separable type II<sub>1</sub> factor  $M$  contains an irreducible hyperfinite factor. Since hyperfinite factor is generated by an operator  $T$ , it follows that  $T$  is an irreducible operator in  $M$ .  $\square$

Recall that an operator  $T$  in  $B(H)$  is a *strongly irreducible operator* if there is no nontrivial idempotents in  $\{T\}'$ . Strongly irreducible operators are generalizations of Jordan blocks in matrix algebras. A rich theory has been set up on this class of operators in the past twenty years (see [19, 20]). Let  $M$  be a type II<sub>1</sub> factor. An operator  $T \in M$  is called a *strongly irreducible operator relative to  $M$*  if  $\{T\}' \cap M = \mathbb{C}1$ . In this section we will give explicit examples of strongly irreducible operators in hyperfinite II<sub>1</sub> factors.

Let  $A_\theta$  be the universal irrational rotation  $C^*$ -algebra with two unitary generators  $u, v$  such that  $vu = e^{2\pi i\theta}uv$ . Then there exists a unique trace  $\tau$  on  $A_\theta$ . Applying the GNS-construction to  $\tau$ , we may assume that  $A_\theta$  acts on  $L^2(A_\theta, \tau)$ . Let  $R$  be the strong operator closure of  $A_\theta$ . Then  $R$  is the hyperfinite type II<sub>1</sub> factor with a unique trace  $\tau$ . Recall that  $u, v$  in  $R$  satisfy the following properties:

1.  $\tau(u^n) = \tau(v^n) = 0$  for all integers  $n \neq 0$ ;
2.  $vu = e^{2\pi i\theta}uv$ ;
3.  $\{u^m v^n : m, n \in \mathbb{Z}\}$  is an orthonormal basis of  $L^2(R) = L^2(R, \tau)$ , where  $u^m v^n$  is viewed as an element of  $L^2(R)$ .

The following theorem is the main result of this section.

**Theorem 10.2.** *For every irrational number  $\theta \in (0, 1)$ ,  $u+v$  is a strongly irreducible operator relative  $R$ , i.e., there exists no nontrivial idempotents in  $\{u+v\}' \cap R$ .*

*Proof.* Let  $x \in \{u+v\}' \cap R$ . By condition 3 above Theorem 10.2,  $x = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n$  and  $\sum_{m,n \in \mathbb{Z}} |\alpha_{m,n}|^2 = \tau(x^*x) < \infty$ . By condition 2 above Theorem 10.2,

$$(u+v)x = (u+v) \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^{m+1} v^n + \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} e^{2\pi i m \theta} u^m v^{n+1} \quad (10.1)$$

and

$$x(u+v) = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n (u+v) = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} e^{2\pi i n \theta} u^{m+1} v^n + \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^{n+1}. \quad (10.2)$$

By condition 3 above Theorem 10.2,  $\{u^m v^n : m, n \in \mathbb{Z}\}$  is an orthonormal basis of  $L^2(R)$ . Comparing the coefficients of the term  $u^m v^n$  in (10.1) and (10.2), we have

$$\alpha_{m-1,n} + \alpha_{m,n-1} e^{2\pi i m \theta} = \alpha_{m-1,n} e^{2\pi i n \theta} + \alpha_{m,n-1},$$

which is equivalent to

$$\alpha_{m-1,n} (1 - e^{2\pi i n \theta}) = \alpha_{m,n-1} (1 - e^{2\pi i m \theta}). \quad (10.3)$$

Since  $\theta$  is an irrational number,  $1 - e^{2\pi i k \theta} \neq 0$  for  $k \neq 0$ . Let  $n = 0$  in equation (10.3). We have  $\alpha_{m,-1} = 0$  for  $m \neq 0$ . Let  $n = -1$  in equation (10.3). We have  $\alpha_{m,-2} = 0$  for  $m \neq 0$ ,  $m \neq 1$ . In general, let  $n = -k$  in equation (10.3). We have  $\alpha_{m,-k-1} = 0$  for  $m \neq 0$ ,  $\dots$ ,  $m \neq k$ . On the other hand, let  $m = 0$  in equation (10.3). We have  $\alpha_{-1,n} = 0$  for  $n \neq 0$ . Similarly, in general we have  $\alpha_{-k-1,n} = 0$  for  $n \neq 0$ ,  $\dots$ ,  $n \neq k$ . So we have  $\alpha_{m,n} = 0$  if either both  $m < 0$  and  $n < 0$  or  $m = -n \neq 0$ .

The motivation of the following part is to prove that  $\alpha_{m,n} = 0$  if either  $m < 0$  or  $n < 0$ . We only need to show that  $\alpha_{m,-k-m} = 0$  and  $\alpha_{-k-m,m} = 0$  for  $k \geq 1$  and  $m \geq 0$ . Repeat using equation (10.3), we have

$$\alpha_{m,-k-m} = \alpha_{0,-k} \frac{1 - e^{-2\pi i k \theta}}{1 - e^{2\pi i \theta}} \cdot \frac{1 - e^{-2\pi i (k+1) \theta}}{1 - e^{2\pi i 2\theta}} \cdots \frac{1 - e^{-2\pi i (k+m-1) \theta}}{1 - e^{2\pi i m \theta}} \quad (10.4)$$

and

$$\alpha_{-k-m,m} = \alpha_{-k,0} \frac{1 - e^{-2\pi i k \theta}}{1 - e^{2\pi i \theta}} \cdot \frac{1 - e^{-2\pi i (k+1)\theta}}{1 - e^{2\pi i 2\theta}} \cdots \frac{1 - e^{-2\pi i (k+m-1)\theta}}{1 - e^{2\pi i m \theta}} \quad (10.5)$$

for  $m \geq 0$  and  $k \geq 1$ .

Let  $k = 1$  in equation (10.4). We have

$$|\alpha_{m,-1-m}| = |\alpha_{0,-1}| \frac{|1 - e^{-2\pi i \theta}|}{|1 - e^{2\pi i \theta}|} \cdot \frac{|1 - e^{-2\pi i 2\theta}|}{|1 - e^{2\pi i 2\theta}|} \cdots \frac{|1 - e^{-2\pi i k \theta}|}{|1 - e^{2\pi i m \theta}|} = |\alpha_{0,-1}|. \quad (10.6)$$

In general for  $k \geq 2$  and  $m \geq 0$ ,

$$|\alpha_{m,-k-m}| = |\alpha_{0,-k}| \frac{|1 - e^{-2\pi i (m+1)\theta}|}{|1 - e^{2\pi i \theta}|} \cdot \frac{|1 - e^{-2\pi i (m+2)\theta}|}{|1 - e^{2\pi i 2\theta}|} \cdots \frac{|1 - e^{-2\pi i (m+k-1)\theta}|}{|1 - e^{2\pi i (k-1)\theta}|}. \quad (10.7)$$

To prove  $\alpha_{0,-k} = 0$  and therefore  $\alpha_{m,-k-m} = 0$  (by equation (10.4)), we use the following fact:

$$\sum_{m,n} |\alpha_{m,n}|^2 < +\infty \Rightarrow \sum_{m>0} |\alpha_{m,-k-m}|^2 < +\infty \Rightarrow \lim_{m \rightarrow +\infty} |\alpha_{m,-k-m}| = 0. \quad (10.8)$$

If  $k = 1$ , then  $|\alpha_{m,-1-m}| = |\alpha_{0,-1}|$  by (10.6). By (10.8), we have  $|\alpha_{m,-1-m}| = |\alpha_{0,-1}| = 0$  for all  $m \geq 0$ .

To prove the general case, we need to use a property of irrational rotation. Namely, there exists a sequence of increasing integers  $m_n$  such that

$$\lim_{n \rightarrow +\infty} e^{2\pi i m_n \theta} = 1.$$

Now for each fixed  $k \geq 2$ , by (10.8) and equation (10.7),

$$\begin{aligned} 0 &= \lim_{n \rightarrow +\infty} |\alpha_{m_n, -k-m_n}| \\ &= \lim_{n \rightarrow +\infty} |\alpha_{0,-k}| \frac{|1 - e^{-2\pi i (m_n+1)\theta}|}{|1 - e^{2\pi i \theta}|} \cdot \frac{|1 - e^{-2\pi i (m_n+2)\theta}|}{|1 - e^{2\pi i 2\theta}|} \cdots \frac{|1 - e^{-2\pi i (m_n+k-1)\theta}|}{|1 - e^{2\pi i (k-1)\theta}|} \\ &= \lim_{n \rightarrow +\infty} |\alpha_{0,-k}| \frac{|e^{2\pi i m_n \theta} - e^{-2\pi i \theta}|}{|1 - e^{2\pi i \theta}|} \cdot \frac{|e^{2\pi i m_n \theta} - e^{-2\pi i 2\theta}|}{|1 - e^{2\pi i 2\theta}|} \cdots \frac{|e^{2\pi i m_n \theta} - e^{-2\pi i (k-1)\theta}|}{|1 - e^{2\pi i (k-1)\theta}|} \\ &= |\alpha_{0,-k}| \frac{|1 - e^{-2\pi i \theta}|}{|1 - e^{2\pi i \theta}|} \cdot \frac{|1 - e^{-2\pi i 2\theta}|}{|1 - e^{2\pi i 2\theta}|} \cdots \frac{|1 - e^{-2\pi i (k-1)\theta}|}{|1 - e^{2\pi i (k-1)\theta}|} \\ &= |\alpha_{0,-k}|. \end{aligned}$$

By equation (10.7),  $|\alpha_{m,-k-m}| = |\alpha_{0,-k}| = 0$  for all  $m \geq 0$  and  $k \geq 1$ . By equation (10.5) and similar arguments,  $|\alpha_{-k-m,m}| = |\alpha_{-k,0}| = 0$  for all  $m \geq 0$  and  $k \geq 1$ .

Above all, we have proved that  $\alpha_{m,n} = 0$  if either  $m < 0$  or  $n < 0$ . Hence

$$x = \sum_{m \geq 0, n \geq 0} \alpha_{m,n} u^m v^n.$$

For  $k \geq 0$ , let  $x_k = \sum_{m \geq 0, n \geq 0, m+n=k} \alpha_{m,n} u^m v^n$ . Then  $x = \sum_{k=0}^{\infty} x_k$  as a vector in  $L^2(R)$ . Since  $x \in \{u + v\}' \cap R$  and  $\{u^m v^n : m, n \in \mathbb{Z}\}$  is an orthonormal basis of  $L^2(R)$ ,  $x_k \in \{u + v\}' \cap R$ . By

equation (10.3),  $\alpha_{m,k-m}$  is uniquely determined by  $\alpha_{0,k}$  for  $0 \leq k \leq m$ . Since  $(u+v)^k$  commutes with  $u+v$ ,  $x_k = \lambda_k(u+v)^k$  for some complex number  $\lambda_k$ . This implies that  $x = \sum_{k=0}^{\infty} \lambda_k(u+v)^k$  and the decomposition is unique.

Suppose  $x \in \{u+v\}' \cap R$ . Let  $x^2 = \sum_{k=0}^{\infty} \sigma_k(u+v)^k$ . For  $a, b \in R$ , let  $\langle a, b \rangle = \tau(b^*a)$ . Then

$$\sigma_k = \langle x^2, u^k \rangle = \langle x, x^* u^k \rangle = \left\langle \sum_{j=0}^{\infty} \lambda_j(u+v)^j, \sum_{j=0}^{\infty} \bar{\lambda}_j ((u+v)^*)^j u^k \right\rangle = \sum_{j=0}^k \lambda_j \lambda_{k-j}, \quad \forall k \geq 0. \quad (10.9)$$

If  $x^2 = x$ , then  $\lambda_k = \sigma_k$  for all  $k$ . Let  $k = 0$ . Then (10.9) implies that  $\lambda_0 = \lambda_0^2$ . So  $\lambda_0 = 0$  or  $\lambda_0 = 1$ . By considering  $1 - x$ , we may assume that  $\lambda_0 = 0$ . Let  $k = 1$ . Then (10.9) implies that  $\lambda_1 = \lambda_0 \cdot \lambda_1 + \lambda_1 \cdot \lambda_0 = 0$ . By (10.9) and induction, we have  $\lambda_k = 0$  for all  $k \geq 0$ . This implies that  $x = 0$ , which completes the proof.  $\square$

By the Riesz spectral decomposition theorem, we immediately have the following corollary.

**Corollary 10.3.** *For every irrational number  $\theta \in (0, 1)$ , the spectrum of  $u + v$  is connected.*

**Remark 10.4.** By the proof of Theorem 10.2, every operator in the commutant algebra of  $u + v$  can be written as a formal series  $\sum_{n=0}^{\infty} a_n(u+v)^n$ . A similar argument can show that for  $0 < \lambda < 1$ , every operator in the commutant algebra of  $u + \lambda v$  can be written as a formal series  $\sum_{n=-\infty}^{\infty} a_n(u + \lambda v)^n$ .

In the following, we will construct more examples of strongly irreducible operators relative to the hyperfinite type  $\text{II}_1$  factor. Precisely, we will prove the following result.

**Proposition 10.5.** *For  $\theta$  in a second category subset of  $[0, 1]$ , we have  $u + v^k$  is strongly irreducible relative to  $R$  for all  $k = 1, 2, \dots$ .*

To prove Proposition 10.5, we need the following lemma.

**Lemma 10.6.** *Let*

$$f_{s,r,k}(z) = \prod_{t=1}^s \frac{1 - z^{kt+r}}{1 - z^{kt}}, \quad E_{r,k} = \{z \in \mathbb{T} : \lim_{s \rightarrow +\infty} f_{s,r,k}(z) = 0\},$$

where  $k$  and  $r$  are positive integers such that  $k \geq 2$  and  $r \not\equiv 0 \pmod{k}$ . Then  $E_{r,k}$  is a first category subset of  $\mathbb{T}$ .

*Proof.* Let  $\epsilon > 0$ . Note that  $f_{s,r,k}(z)$  is a meromorphic function with finite poles on  $\mathbb{T}$ . So the set

$$D_{s,r,k,\epsilon} \triangleq \{z \in \mathbb{T} : |f_{s,r,k}(z)| \leq \epsilon\}$$

is a closed subset of  $\mathbb{T}$ . Let

$$E_{s,r,k,\epsilon} \triangleq \{z \in \mathbb{T} : |f_{s,r,k}(z)| \leq \epsilon, \forall a \geq s\}.$$

Then  $E_{s,r,k,\epsilon} = \bigcap_{a \geq s} D_{a,r,k,\epsilon}$  is also a closed subset of  $\mathbb{T}$ .

Let  $F_{s,r,k,\epsilon} = \mathbb{T} \setminus E_{s,r,k,\epsilon}$ . Then  $F_{s,r,k,\epsilon}$  is an open subset of  $\mathbb{T}$ , and

$$\begin{aligned} F_{s,r,k,\epsilon} &= \mathbb{T} \setminus \bigcap_{a \geq s} D_{a,r,k,\epsilon} \\ &= \bigcup_{a \geq s} (\mathbb{T} \setminus D_{a,r,k,\epsilon}) \\ &= \bigcup_{a \geq s} \{z \in \mathbb{T} : |f_{a,r,k}(z)| > \epsilon\} \\ &\supseteq \bigcup_{a \geq s} \{\text{poles of } f_{a,r,k}(z)\} \\ &= \bigcup_{a \geq s} \{z : z^{ak} = 1\}. \end{aligned}$$

So  $F_{s,r,k,\epsilon}$  is a dense open subset of  $\mathbb{T}$ , which implies that  $E_{s,r,k,\epsilon}$  is a nowhere dense closed subset of  $\mathbb{T}$ . Therefore,  $E_{r,k} \subseteq \bigcup_{s=1}^{\infty} E_{s,r,k,\epsilon}$  is a first category subset of  $\mathbb{T}$ .  $\square$

*Proof of Proposition 10.5:* Define  $f_{s,r,k}(z)$  and  $E_{r,k}$  as in Lemma 10.6. By Lemma 10.6,  $E_{r,k}$  is a first category subset of  $\mathbb{T}$ . So  $\bigcup_{r \neq 0 \pmod{k}} E_{r,k}$  is also a first category subset of  $\mathbb{T}$ . Hence

$$\mathbb{T} \setminus \{e^{2\pi i\theta} : \theta \in [0, 1] \text{ is an rational number}\} \setminus \bigcup_{r \neq 0 \pmod{k}} E_{r,k}$$

is a second category subset of  $\mathbb{T}$ . Choose a  $\theta \in [0, 1]$  such that  $e^{2\pi i\theta}$  is in the above set. Then for all  $r$  with  $r \neq 0 \pmod{k}$ ,

$$\lim_{s \rightarrow +\infty} f_{s,r,k}(z) = 0$$

does not hold.

Let  $x \in \{u+v^k\}' \cap R$  be an idempotent. By condition 3 above Theorem 10.2,  $x = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n$  and  $\sum_{m,n \in \mathbb{Z}} |\alpha_{m,n}|^2 = \tau(x^* x) < \infty$ . By condition 2 above Theorem 10.2,

$$(u+v^k)x = (u+v^k) \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^{m+1} v^n + \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} e^{2\pi i k m \theta} u^m v^{n+k} \quad (10.10)$$

and

$$x(u+v^k) = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^n (u+v) = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} e^{2\pi i n \theta} u^{m+1} v^n + \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} u^m v^{n+k}. \quad (10.11)$$

By condition 3 above Theorem 10.2,  $\{u^m v^n : m, n \in \mathbb{Z}\}$  is an orthonormal of  $L^2(R)$ . Comparing the coefficients of the term  $u^m v^n$  in (10.10) and (10.11), we have

$$\alpha_{m-1,n} + \alpha_{m,n-k} e^{2\pi i(km)\theta} = \alpha_{m-1,n} e^{2\pi i n \theta} + \alpha_{m,n-k},$$

which is equivalent to

$$\alpha_{m-1,n}(1 - e^{2\pi i n \theta}) = \alpha_{m,n-k}(1 - e^{2\pi i (km)\theta}). \quad (10.12)$$

Since  $\theta$  is an irrational number,  $1 - e^{2\pi i k \theta} \neq 0$  for  $k \neq 0$ . Let  $n = 0$  in equation (10.12). We have  $\alpha_{m,-k} = 0$  for  $m \neq 0$ . Let  $n = -k$  in equation (10.12). We have  $\alpha_{m,-2k} = 0$  for  $m \neq 0, m \neq 1$ . In general, let  $n = -sk$  in equation (10.12). We have  $\alpha_{m,-(s+1)k} = 0$  for  $m \neq 0, \dots, m \neq s$ . On the other hand, let  $m = 0$  in equation (10.12). We have  $\alpha_{-1,n} = 0$  for  $n \neq 0$ . Similarly, in general we have  $\alpha_{-s-1,n} = 0$  for  $n \neq 0, k, \dots, sk$ .

Claim that  $\alpha_{m,n} = 0$  if either  $m < 0$  or  $n < 0$ . By the above arguments, we only need to show that  $\alpha_{0,-r} = 0$  and  $\alpha_{-r,0} = 0$  for  $r \geq 1$ . Firstly, we show that  $\alpha_{-r,0} = 0$  for  $r \geq 1$ .

In equation (10.12), let  $m = -r$  and  $n = k$ . We have

$$\alpha_{-r-1,k}(1 - e^{2\pi i (k)\theta}) = \alpha_{-r,0}(1 - e^{-2\pi i (kr)\theta}). \quad (10.13)$$

In equation (10.12), let  $m = -r - 1$  and  $n = 2k$ . We have

$$\alpha_{-r-2,2k}(1 - e^{2\pi i (2k)\theta}) = \alpha_{-r-1,k}(1 - e^{-2\pi i (k(r+1))\theta}). \quad (10.14)$$

In general, for a positive integer  $s$ , let  $m = -r - s + 1$  and  $n = (s - 1)k$  in equation (10.12). We have

$$\alpha_{-r-s,sk}(1 - e^{2\pi i (sk)\theta}) = \alpha_{-r-s+1,k}(1 - e^{-2\pi i (k(r+s-1))\theta}). \quad (10.15)$$

By equations (10.13), (10.14), (10.15),

$$\alpha_{-r-s,sk} = \alpha_{-r,0} \prod_{t=1}^s \frac{1 - e^{-2\pi i (k(r+s-t))\theta}}{1 - e^{2\pi i (tk)\theta}}.$$

So for  $s > r - 1$ , we have

$$|\alpha_{-r-s,sk}| = |\alpha_{-r,0}| \prod_{t=1}^{r-1} \frac{|1 - e^{-2\pi i ((r+s))k\theta} e^{2\pi i (tk)\theta}|}{|1 - e^{2\pi i (tk)\theta}|}. \quad (10.16)$$

Since  $\theta$  is an irrational number, there is a sequence positive integers  $s_n$  such that

$$\lim_{n \rightarrow \infty} e^{-2\pi i (r+s_n)k\theta} = 1.$$

By equation (10.16),  $\lim_{n \rightarrow \infty} |\alpha_{-r-s_n,s_n k}| = |\alpha_{-r,0}|$ . Since  $\sum_{n=1}^{\infty} |\alpha_{-r-s_n,s_n k}|^2 < \infty$ ,  $|\alpha_{-r,0}| = \lim_{n \rightarrow \infty} |\alpha_{-r-s_n,s_n k}| = 0$ .

Secondly, we show that  $\alpha_{0,-r} = 0$  for  $r \geq 1$ . In equation (10.12), let  $m = 1$  and  $n = -r$ . We have

$$\alpha_{0,-r}(1 - e^{2\pi i (-r)\theta}) = \alpha_{1,-r-k}(1 - e^{2\pi i (k)\theta}). \quad (10.17)$$



In equation (10.12), let  $m = 2$  and  $n = -r - k$ . We have

$$\alpha_{1,-r-k}(1 - e^{2\pi i(-r-k)\theta}) = \alpha_{2,-r-2k}(1 - e^{2\pi i(2k)\theta}). \quad (10.18)$$

In general, for a positive integer  $s$ , let  $m = s + 1$  and  $n = -r - (s - 1)k$  in equation (10.12). We have

$$\alpha_{0,-r-(s-1)k}(1 - e^{2\pi i(-r-(s-1)k)\theta}) = \alpha_{s,-r-sk}(1 - e^{2\pi i(sk)\theta}). \quad (10.19)$$

By equations (10.17), (10.18), (10.19),

$$\alpha_{s,-r-sk} = \alpha_{0,-r} \prod_{t=1}^s \frac{1 - e^{-2\pi i(tk+(r-k))\theta}}{1 - e^{2\pi i(tk)\theta}}. \quad (10.20)$$

We consider two cases. Case 1:  $r = 0(\text{mod } k)$ . By equation (10.20),

$$\alpha_{s,-r-sk} = \alpha_{0,-r} \prod_{t=1}^{\frac{r-k}{k}} \frac{1 - e^{-2\pi i(sk+r)\theta} e^{2\pi i(tk)\theta}}{1 - e^{2\pi i(tk)\theta}} \quad (10.21)$$

for  $s > \frac{r-k}{k}$ . Since  $\theta$  is an irrational number, there is a sequence positive integers  $s_n$  such that

$$\lim_{n \rightarrow \infty} e^{-2\pi i(s_n k + r)\theta} = 1.$$

By equation (10.21),  $\lim_{n \rightarrow \infty} |\alpha_{s_n, -r-s_n k}| = |\alpha_{0,-r}|$ . Since  $\sum_{n=1}^{\infty} |\alpha_{s_n, -r-s_n k}|^2 < \infty$ ,

$$|\alpha_{0,-r}| = \lim_{n \rightarrow \infty} |\alpha_{s_n, -r-s_n k}| = 0.$$

Case 2:  $r \neq 0(\text{mod } k)$ . Note that  $\sum_{s=1}^{\infty} |\alpha_{s,-r-sk}|^2 < \infty$ . So  $\lim_{s \rightarrow \infty} \alpha_{s,-r-sk} = 0$ . By the choice of  $\theta$ ,

$$\lim_{s \rightarrow \infty} \prod_{t=1}^s \frac{1 - e^{-2\pi i(tk+(r-k))\theta}}{1 - e^{2\pi i(tk)\theta}} = 0$$

does not hold. So  $\alpha_{0,-r}$  has to be 0.

Above all, we have proved that  $\alpha_{m,n} = 0$  if either  $m < 0$  or  $n < 0$ . Furthermore, we claim that  $\alpha_{m,n} = 0$  for  $m, n \geq 0$  and  $n \neq 0(\text{mod } k)$ . Let  $s$  be the least positive integer greater than  $n/k$ . By equation (10.12), we have

$$\begin{aligned} \alpha_{m,n}(1 - e^{2\pi i n \theta}) &= \alpha_{m+1,n-k}(1 - e^{2\pi i k(m+1)\theta}), \\ \alpha_{m+1,n-k}(1 - e^{2\pi i(n-k)\theta}) &= \alpha_{m+2,n-2k}(1 - e^{2\pi i k(m+2)\theta}), \\ &\vdots \\ \alpha_{m+s-1,n-(s-1)k}(1 - e^{2\pi i(n-(s-1)k)\theta}) &= \alpha_{m+s,n-sk}(1 - e^{2\pi i k(m+s)\theta}). \end{aligned}$$

Since  $n - sk < 0$ ,  $\alpha_{m+s,n-sk} = 0$ . The above equations imply that  $\alpha_{m,n} = 0$  since  $1 - e^{2\pi i(n-jk)\theta} \neq 0$  for all  $j$ .

Hence

$$x = \sum_{m \geq 0, n \geq 0} \alpha_{m,n} u^m v^{kn},$$

which implies that  $x$  is in the commutant algebra of  $u + v^k$  relative to the von Neumann subalgebra generated by  $u$  and  $v^k$ . Since  $v^k u = e^{2\pi i k \theta} v^k u$  and  $k\theta$  is an irrational number,  $x = 0$  or  $x = 1$  by Theorem 10.2. So  $T$  is a strongly irreducible operator relative to  $R$ . This completes the proof of Theorem 10.5.

**Proposition 10.7.** *Let  $n$  be a positive integer. Then by Theorem 8.2  $N = W^*(u + v^n) = W^*(u, v^n)$  is an irreducible subfactor of  $W^*(u + v) = R$  with Jones index [21]  $[R : N] = n$ .*

*Proof.* Since  $R$  is generated by  $u, v$  and  $N$  is generated by  $u, v^n$ , it is clear that every element of  $R$  can be written as finite linear combinations of elements in  $Nv^i, 0 \leq i \leq n-1$ . Since  $Nv^i$  is orthogonal to  $Nv^j, 0 \leq i \neq j \leq n-1$ , under the inner product defined by the trace on  $R$ , it follows that  $R = N \oplus Nv \oplus Nv^2 \oplus \cdots \oplus Nv^{n-1}$ , where  $Nv^i$  is orthogonal to  $Nv^j, 0 \leq i \neq j \leq n-1$ . So by [30],  $v^i, 0 \leq i \leq n-1$  is a Pimsner-Popa basis of  $R$  over  $N$ , and since  $v^i$  is unitary, it follows that  $[R : N] = n$ .  $\square$

On the other hand, by Proposition 10.5 for  $\theta$  in a second category subset of  $[0, 1]$ ,  $u + v^n$  is strongly irreducible relative to  $R$ . So for every bounded invertible operator  $x \in R$ ,  $x(u + v^n)x^{-1}$  generates an irreducible subfactor  $W^*(x(u + v^n)x^{-1})$  of  $R$ . Is it true that  $[R : W^*(x(u + v^n)x^{-1})] = n$  for all bounded invertible operators  $x$  in  $R$ , at least when  $x$  is close to identity in norm?

By definitions if  $T$  is strongly irreducible relative to  $M$ , then  $T$  is irreducible relative to  $M$ . An operator  $T$  is strongly irreducible relative to a type  $\text{II}_1$  factor if and only if  $XTX^{-1}$  is an irreducible operator relative to  $M$  for every bounded invertible operator  $X \in M$ . However, if  $T$  is irreducible relative to  $M$ , this is not true in general. The following result shows that an irreducible operator relative to  $M$  can be similar to a unitary operator.

**Proposition 10.8.** *Let  $\theta$  be an irrational number in  $[0, 1]$  and let  $n$  be any positive integer. Then in the hyperfinite type  $\text{II}_1$  factor  $R$  there exists a bounded invertible operator  $x$  such that  $W^*(xux^{-1}) = W^*(u + v^n) = W^*(u, v^n)$ .*

*Proof.* Let  $\sigma$  be a nonempty open connected subset of  $\sigma(v^n) = \mathbb{T}$  such that  $\sigma \cap e^{2\pi i n \theta} \sigma = \emptyset$ . Let  $x = 1 - \frac{E_{v^n}(\sigma)}{2} \in R$ , where  $E_{v^n}(\cdot)$  denotes the spectral measure of  $v^n$ . Since  $v^n u = e^{2\pi i n \theta} u v^n$ ,  $f(v^n)u = u f(e^{2\pi i n \theta} v)$  for all  $f \in L^\infty(\mathbb{T}, m)$ . Therefore,

$$E_{v^n}(\sigma)u = u E_{v^n}(e^{2\pi i n \theta} \sigma), \quad (10.22)$$

which implies that

$$u^{-1}E_{v^n}(\sigma) = E_{v^n}(e^{2\pi i n\theta}\sigma)u^{-1}. \quad (10.23)$$

Similarly, by  $v^n u^{-1} = e^{-2\pi i n\theta} u^{-1} v^n$ , we have

$$uE_{v^n}(\sigma) = E_{v^n}(e^{-2\pi i n\theta}\sigma)u. \quad (10.24)$$

Combining the above equations, we have

$$(xux^{-1})^*(xux^{-1}) = x^{-1}u^{-1}xxux^{-1} = \left(1 - \frac{E_{v^n}(\sigma)}{2}\right)^{-1} \left(1 - \frac{E_{v^n}(e^{2\pi i n\theta}\sigma)}{2}\right)^2 \left(1 - \frac{E_{v^n}(\sigma)}{2}\right)^{-1}.$$

We can write

$$1 - \frac{E_{v^n}(e^{2\pi i n\theta}\sigma)}{2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \text{Ran } E_{v^n}(\mathbb{T} \setminus (e^{2\pi i n\theta}\sigma \cup \sigma)) \\ \text{Ran } E_{v^n}(e^{2\pi i n\theta}\sigma) \\ \text{Ran } E_{v^n}(\sigma) \end{matrix}$$

and write

$$1 - \frac{E_{v^n}(\sigma)}{2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{matrix} \text{Ran } E_{v^n}(\mathbb{T} \setminus (e^{2\pi i n\theta}\sigma \cup \sigma)) \\ \text{Ran } E_{v^n}(e^{2\pi i n\theta}\sigma) \\ \text{Ran } E_{v^n}(\sigma) \end{matrix}.$$

So we have

$$(xux^{-1})^*(xux^{-1}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{matrix} \text{Ran } E_{v^n}(\mathbb{T} \setminus (e^{2\pi i n\theta}\sigma \cup \sigma)) \\ \text{Ran } E_{v^n}(e^{2\pi i n\theta}\sigma) \\ \text{Ran } E_{v^n}(\sigma) \end{matrix} \in W^*(xux^{-1}).$$

Therefore,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{matrix} \text{Ran } E_{v^n}(\mathbb{T} \setminus (e^{2\pi i n\theta}\sigma \cup \sigma)) \\ \text{Ran } E_{v^n}(e^{2\pi i n\theta}\sigma) \\ \text{Ran } E_{v^n}(\sigma) \end{matrix} \in W^*(xux^{-1}).$$

This implies that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \text{Ran } E_{v^n}(\mathbb{T} \setminus (e^{2\pi i n\theta}\sigma \cup \sigma)) \\ \text{Ran } E_{v^n}(e^{2\pi i n\theta}\sigma) \\ \text{Ran } E_{v^n}(\sigma) \end{matrix} \in W^*(xux^{-1}).$$

Note that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}^k \xrightarrow{\text{SOT}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} \text{Ran } E_{v^n}(\mathbb{T} \setminus (e^{2\pi i n\theta}\sigma \cup \sigma)) \\ \text{Ran } E_{v^n}(e^{2\pi i n\theta}\sigma) \\ \text{Ran } E_{v^n}(\sigma) \end{matrix} = E_{v^n}(\sigma).$$

Hence,  $E_{v^n}(\sigma) \in W^*(xux^{-1})$  and  $x = 1 - \frac{E_{v^n}(\sigma)}{2} \in W^*(xux^{-1})$ . Therefore,  $u \in W^*(xux^{-1})$ . Note that

$$E_{v^n}(e^{2\pi i k n \theta} \sigma) = u^{-k} E_{v^n}(\sigma) u^k \in W^*(xux^{-1}), \forall k \in \mathbb{N}.$$

Since  $\{e^{2\pi i k n \theta}\}_{k=1}^\infty$  is dense in  $\mathbb{T}$ ,  $E_{v^n}(\sigma_1) \in W^*(xux^{-1})$  for every open connected subset  $\sigma_1$  in  $\mathbb{T}$  which has same arc length as  $\sigma$ . If  $\sigma_0$  is an open connected subset of  $\mathbb{T}$  with arc length smaller than the arc length of  $\sigma$ , then there are two open connected subsets  $\sigma_1, \sigma_2$  of  $\mathbb{T}$  with arc length same as  $\sigma$  such that  $\sigma_1 \cap \sigma_2 = \sigma_0$ . Thus

$$E_{v^n}(\sigma_0) = E_{v^n}(\sigma_1) \cap E_{v^n}(\sigma_2) \in W^*(xux^{-1}).$$

This implies that for every measurable subset  $F$  of  $\mathbb{T}$ , we have  $E_{v^n}(F) \in W^*(xux^{-1})$ . So  $v^n \in W^*(xux^{-1})$  and we have proved that  $W^*(xux^{-1}) = W^*(u, v^n)$ .  $\square$

In general, we have the following observation.

**Proposition 10.9.** *Let  $N \subseteq M$  be an inclusion of type  $\text{II}_1$  factors. Then there exists an operator  $S \in N$  which is similar to an irreducible operator  $T$  relative to  $M$ .*

*Proof.* We may identify  $N = M_3(\mathbb{C}) \otimes N_1$  and  $M = M_3(\mathbb{C}) \otimes M_1$ . Choose complex numbers  $\alpha_1, \alpha_2, \alpha_3$  such that  $\alpha_i \neq \alpha_j$  for  $i \neq j$ . Let  $D$  be an irreducible operator in  $M_1$ ,

$$S = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \quad T = \begin{pmatrix} \alpha_1 & 1 & D \\ 0 & \alpha_2 & 1 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & \frac{1}{\alpha_1 - \alpha_2} & \frac{1}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)} + \frac{D}{\alpha_1 - \alpha_3} \\ 0 & 1 & \frac{1}{\alpha_2 - \alpha_3} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then direct calculations show that  $T$  is an irreducible operator in  $M$  and  $XSX^{-1} = T$ .  $\square$

## 11 Spectrum of $u + \lambda v$

**Theorem 11.1.** *For every irrational number  $\theta \in (0, 1)$ ,*

$$\sigma(u + \lambda v) = \begin{cases} \mathbb{T} & 0 < \lambda < 1, \\ \overline{B(0, 1)} & \lambda = 1, \\ \lambda \mathbb{T} & \lambda > 1, \end{cases}$$

where  $\mathbb{T}$  is the unit circle.

*Proof.* Note that  $u + v = u(1 + u^*v)$ . Since  $u^*v$  is a Haar unitary operator,  $-1 \in \sigma(u^*v)$ . This implies that  $u + v$  is not invertible and therefore  $0 \in \sigma(u + v)$ . For every  $\theta \in [0, 2\pi]$ ,  $e^{i\theta}u$  and  $e^{i\theta}v$  satisfy the

same irrational rotation relation as  $u$  and  $v$ , so  $\sigma(u + v)$  is rotation symmetric with respect to 0. By Corollary 10.3,  $\sigma(u + v)$  is a closed disk with center 0. By Lemma 9.5,  $\sigma(u + v) = \overline{B(0, 1)}$ .

For  $0 < \lambda < 1$ ,  $u + \lambda v = u(1 + \lambda u^* v)$  is invertible. In the following we prove that  $r((u + \lambda v)^{-1}) \leq 1$ . The proof is similar to the proof of Lemma 9.5. However, some details should be treated carefully, so we include the complete proof. By equation (9.1),

$$(u + \lambda v)^{-n} = (1 + \lambda w)^{-1} (1 + \alpha \lambda w)^{-1} \cdots (1 + \alpha^{n-1} \lambda w)^{-1} u^{-n}, \quad \forall n \in \mathbb{N}.$$

Hence,

$$\begin{aligned} \|(u + \lambda v)^{-n}\|^{1/n} &= \|(1 + \lambda w)^{-1} (1 + \alpha \lambda w)^{-1} \cdots (1 + \alpha^{n-1} \lambda w)^{-1}\|^{1/n} \\ &= \left( \max_{z \in \mathbb{T}} |(1 + \lambda z)^{-1} (1 + \alpha \lambda z)^{-1} \cdots (1 + \alpha^{n-1} \lambda z)^{-1}| \right)^{1/n} \\ &= \max_{x \in [0, 1]} \left( \prod_{k=0}^{n-1} (1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta)))^{-1} \right)^{\frac{1}{2n}}. \end{aligned}$$

Let  $\epsilon > 0$ . Note that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{2n} \left( \sum_{k=1}^n -\ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{n} \right) \right) + \sum_{k=n}^{2n-1} -\ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{n} \right) \right) \right) \\ = - \int_0^1 \ln(1 + \lambda^2 + 2\lambda \cos 2\pi x) dx = 0. \end{aligned}$$

There is  $N \in \mathbb{N}$  such that

$$\frac{1}{2N} \left( \sum_{k=1}^N -\ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) + \sum_{k=N}^{2N-1} -\ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \right) < \epsilon/2.$$

Let

$$L(\lambda) = \max_{1 \leq k \leq 2N-1} \left| \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \right|.$$

Then for  $0 < \lambda < 1$ ,  $L(\lambda) < \infty$  (Note that if  $\lambda = 1$ , then  $L(\lambda) = \infty$ ). Divide the unit circle  $\mathbb{T}$  into  $2N$  equal sections  $A_1, \dots, A_{2N}$ . By Lemma 9.4, there exists  $N'$  such that for all  $n \geq N'$  and all  $x \in [0, 1]$ , if  $A_k$  contains  $n/(2N) + r_k(x)$  points of  $e^{2\pi i x}, \alpha e^{2\pi i x}, \dots, \alpha^{n-1} e^{2\pi i x}$ , then  $\frac{\sum_{k=1}^{2N} |r_k(x)|}{n} < \frac{\epsilon}{L(\lambda)}$ . Note that  $\cos 2\pi x$  is decreasing for  $x \in [0, 1/2]$  and increasing for  $x \in [1/2, 1]$ . Therefore, for all  $x \in [0, 1]$ ,

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} -\ln(1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta))) &\leq \frac{1}{n} \sum_{k=1}^N -\left( \frac{n}{2N} + r_k(x) \right) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \\ &\quad + \frac{1}{n} \sum_{k=N}^{2N-1} -\left( \frac{n}{2N} + r_{k+1}(x) \right) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2N} \left( \sum_{k=1}^N -\ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) + \sum_{k=N}^{2N-1} -\ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \right) \\
&+ \frac{1}{n} \sum_{k=1}^N -r_k(x) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) + \frac{1}{n} \sum_{k=N}^{2N-1} -r_{k+1}(x) \ln \left( 1 + \lambda^2 + 2\lambda \cos \left( \frac{k\pi}{N} \right) \right) \\
&< \epsilon + \frac{1}{n} \sum_{k=1}^{2N} |r_k(x)| L(\lambda) < 2\epsilon.
\end{aligned}$$

This implies that for all  $n \geq N'$  and  $x \in [0, 1]$ ,

$$\left( \prod_{k=0}^{n-1} (1 + \lambda^2 + 2\lambda \cos(2\pi(x + k\theta)))^{-1} \right)^{\frac{1}{2n}} \leq e^{2\epsilon}.$$

Therefore,  $\|(u + \lambda v)^{-n}\|^{1/n} \leq e^{2\epsilon}$  for all  $n \geq N'$ . So  $r((u + \lambda v)^{-1}) \leq e^{2\epsilon}$ . Since  $\epsilon > 0$  is arbitrary,  $r((u + \lambda v)^{-1}) \leq 1$ . By Lemma 9.5,  $r(u + \lambda v) = 1$  for  $0 < \lambda < 1$ . This implies that  $\sigma(u + \lambda v) \subseteq \mathbb{T}$ . Since  $\sigma(u + \lambda v)$  is rotation invariant,  $\sigma(u + \lambda v) = \mathbb{T}$ .

If  $\lambda > 1$ , then  $\sigma(u + \lambda v) = \lambda\sigma(\lambda^{-1}u + v) = \lambda\mathbb{T}$ . This completes the proof.  $\square$

## 12 Brown's spectral distribution of $u + \lambda v$

Let  $M$  be a finite von Neumann algebra with a faithful normal tracial state  $\tau$ . The *Fuglede-Kadison determinant* [14],  $\Delta : M \rightarrow [0, +\infty[$ , is given by

$$\Delta(T) = \exp\{\tau(\ln |T|)\}, \quad T \in M,$$

with  $\exp\{-\infty\} := 0$ . For an arbitrary element  $T$  in  $M$  the function  $\lambda \rightarrow \ln \Delta(T - \lambda 1)$  is subharmonic on  $\mathbb{C}$ , and its Laplacian

$$d\mu_T(\lambda) := \frac{1}{2\pi} \nabla^2 \ln \Delta(T - \lambda 1),$$

in the distribution sense, defines a probability measure  $\mu_T$  on  $\mathbb{C}$ , called the *Brown's spectral distribution* or *Brown measure* of  $T$ . From the definition, Brown measure  $\mu_T$  only depends on the joint distribution of  $T$  and  $T^*$ , i.e., the (noncommutative) mixed moments of  $T$  and  $T^*$ .

If  $T$  is normal, then  $\mu_T$  is the trace  $\tau$  composed with the spectral projections of  $T$ . If  $M = M_n(\mathbb{C})$ , then  $\mu_T$  is the normalized counting measure  $\frac{1}{n} (\delta_{\lambda_1} + \delta_{\lambda_2} + \cdots + \delta_{\lambda_n})$ , where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigenvalues of  $T$  repeated according to root multiplicity.

The following theorem is Theorem 2.2 of [17].

**Theorem 12.1.** *Let  $T \in M$ , and for  $n \in \mathbb{N}$ , let  $\mu_n \in \text{Prob}([0, \infty))$  denote the distribution of  $(T^n)^*T^n$  w.r.t  $\tau$ , and let  $\nu_n$  denote the push-forward measure of  $\mu_n$  under the map  $t \rightarrow t^{\frac{1}{n}}$ . Moreover, let  $\nu$  denote the push-forward measure of  $\mu_T$  under the map  $z \rightarrow |z|^2$ , i.e.,  $\nu$  is determined by*

$$\nu([0, t^2]) = \mu_T(\overline{B(0, t)}), \quad , t > 0.$$

*Then  $\nu_n \rightarrow \nu$  weakly in  $\text{Prob}([0, \infty))$ .*

**Theorem 12.2.** *The Brown measure of  $u + \lambda v$  is the Haar measure on the unit circle  $\mathbb{T}$  if  $0 < \lambda \leq 1$  and the Haar measure on  $\lambda\mathbb{T}$  if  $\lambda > 1$ .*

*Proof.* By Theorem 11.1,  $\sigma(u + \lambda v) = \mathbb{T}$  if  $0 < \lambda < 1$  and  $\sigma(u + \lambda v) = \lambda\mathbb{T}$  if  $\lambda > 1$ . Since  $\mu_{(u+\lambda v)}$  is rotation invariant and the support of  $\mu_{(u+\lambda v)}$  is contained in  $\sigma(u + \lambda v)$ , the Brown measure of  $u + \lambda v$  is the Haar measure on the unit circle  $\mathbb{T}$  if  $0 < \lambda < 1$  and the Haar measure on  $\lambda\mathbb{T}$  if  $\lambda > 1$ .

In the following, we consider the case  $\lambda = 1$ . Let  $T = u + v$ , and let  $\nu$  and  $\nu_n$  be the measures defined as in Theorem 12.1. Note that  $((T^n)^*T^n)^{\frac{1}{n}} = |(1 + w) \cdots (1 + \alpha^{n-1}w)|^{\frac{2}{n}}$ , where  $w = u^*v$  is a Haar unitary operator. So we can view  $((T^n)^*T^n)^{\frac{1}{n}}$  as the multiplication operator on  $L^2[0, 1]$  corresponding to the function

$$\left| \prod_{k=0}^{n-1} (2 + 2 \cos(2\pi(x + k\theta))) \right|^{\frac{1}{n}}.$$

Let  $m$  be the Lebesgue measure on  $[0, 1]$ . For  $0 < b < 1$ , since  $[0, b)$  is an open set relative to  $[0, \infty)$  and  $\nu_n \rightarrow \nu$  weakly in  $\text{Prob}([0, \infty))$  (by Theorem 12.1),

$$\nu([0, b)) \leq \liminf_{n \rightarrow \infty} \nu_n([0, b)) = \liminf_{n \rightarrow \infty} m \left( \left\{ x : \left| \prod_{k=0}^{n-1} (2 + 2 \cos(2\pi(x + k\theta))) \right|^{\frac{1}{n}} \in [0, b) \right\} \right).$$

By Lemma 9.2, for almost all  $x \in [0, 1]$ ,

$$\lim_{n \rightarrow \infty} \left| \prod_{k=0}^{n-1} (2 + 2 \cos(2\pi(x + k\theta))) \right|^{\frac{1}{n}} = 1.$$

In particular,  $\left| \prod_{k=0}^{n-1} (2 + 2 \cos(2\pi(x + k\theta))) \right|^{\frac{1}{n}}$  converges in measure to the constant function 1 on  $[0, 1]$ . Since  $b < 1$ ,  $\nu([0, b)) = 0$ . Let  $r'(u + v)$  be the Brown spectral radius of  $u + v$ . Then  $r'(u + v) \leq r(u + v) = 1$  (see [17], Corollary 2.6). So the support of  $\nu$  is contained in  $[0, 1]$ . Thus  $\nu$  is the Dirac measure  $\delta_1$  and the support of  $\mu_T$  is contained in  $\mathbb{T}$ . Since  $\mu_T$  is rotation invariant,  $\mu_T$  is the Haar measure on  $\mathbb{T}$ .  $\square$

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